Unobserved Heterogeneity in Auctions

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Background

- In many applications one would expect bidders to observe relevant information that is not available to researchers.
- Accounting for this unobserved heterogeneity can be important, especially in first-price auctions: identification relies on the econometrician's ability to estimate the conditional probabilities entering bidders' first-order conditions.

Ignoring Unobserved Heterogeneity

Misspecified model can lead to misleading results

- intuitively, we will infer too much...
  - ... within-auction *correlation* in bidders' types
  - ... cross-auction *variation* in bidders' types
- exaggerated market power / information rents to bidders (Krasnokutskaya, 2011; Krasnokutskaya & Seim, 2011; Athey, Levin, & Seira, 2011)
- wrong conclusions about optimal reserve prices (Krasnokutskaya, 2011; Roberts, 2013)
- wrong conclusions in test for common values (Haile, Hong & Shum (2003), Compiani, Haile & Sant'Anna, 2017).

Today's Talk

Focus on first-price auctions

- review established approaches to identification with unobserved heterogeneity
- discuss two new approaches that may be more suitable for some applications.

Model

Setup

First-price sealed bid auction

- sale of single indivisible good
- auction t characteristics $C_t \equiv (X_t, Z_t, U_t)$
  - (X_t, Z_t) observable to econometrician, U_t unobserved
  - (X_t, U_t) affect bidder valuations, Z_t doesn't
- N_t risk-neutral bidders
- (N_t, C_t) common knowledge among bidders
- no reserve price (minimum bid)

(some things assumed here/later for simplicity can be relaxed).

Valuations and Information

Notation and terminology

- bidder *valuations*: $V_t = (V_{1t}, \ldots, V_{n_t t})$
- bidder *signals* (private information): $S_t = (S_{1t}, \ldots, S_{n_t t})$
- a bidder's *pivotal expectation*

$$w(s; n, c) \equiv E\left[V_{it} | S_{it} = max_{j \neq i} S_{jt} = s, N_t = n, C_t = c\right]$$

(expected value of winning | own signal and the event {tied to win})

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special cases:

- $S_{it} = V_{it} \implies$ private values, $w(s; n, c) = s = v$
- $E\left[V_{it} | S_t\right]$ dependent on $S_{-it} \implies$ common values (a.k.a. interdependent values).

Stochastic Structure

Joint distribution $F_{SV}\left(S_t, V_t | n, C_t\right)$, assumed...

- affiliated (positive dependence; independence = special case)
- exchangeable in indices (symmetric bidders)
- admits $\mathcal{C}^1$ density, positive on $(\underline{s}, \overline{s})^n \times (\underline{v}, \overline{v})^n$
- $E[V_{it}|S_{it}, S_{-it}, N_t, X_t, U_t]$ exists, strictly increasing in S_{it}.

Equilibrium Bidding

Bayes Nash equilibrium in pure, strictly increasing, differentiable strategies (see Athey and Haile (2007) re existence, uniqueness results)

- notation:
  - eqm bidding strategy $\beta(\cdot; N_t, C_t) : [\underline{s}, \overline{s}] \to \mathbb{R}$
  - i's bid (action) $B_{it} = \beta(S_{it}; N_t, C_t)$
  - maximum bid among i's competitors: M_{it}
- i's bidding problem:

$$\max_b E\left[(V_{it} - b)\, 1\{M_{it} < b\} \,|s_{it}, n_t, c_t\right].$$

Identification

Observables and structural features of interest

- typical primitives of interest:
  - $F_{SV}(S_t, V_t | n, C_t)$
  - distn $F_w(\cdot | n, c)$ of $w(S_{it}; n, c)$
  - distn of $U_t | X_t, Z_t, N_t$
- observables (for today):
  - all bids, N_t, X_t
  - possibly Z_t
  - NOT U_t
- our problem: bidders condition on U_t, but we cant.

Inverting Equilibrium First-Order Conditions

Guerre, Perrigne & Vuong (2000)

- take the case of private values
  - i knows $V_{it} = v, N_t = n, C_t = c$
  - primitives of interest: $F_V\left(V_{1t}, \ldots, V_{nt} | n, c\right)$
- $G_{M|S}\left(m|b, n, c\right) \equiv \Pr\left(M_{it} \leq m | \beta\left(S_{it}; n, c\right) = b, N_t = n, C_t = c\right)$
- bidder i solves

$$\max_{\tilde{b}} \left(v - \tilde{b}\right) G_{M|B}\left(\tilde{b}|b, n, c\right)$$

with FOC

$$v = b + \frac{G_{M|B}\left(b|b, n, c\right)}{g_{M|B}\left(b|b, n, c\right)}.$$

Inverting Equilibrium FOC (ctd)

More generally (allowing common values), FOC has form

$$w(s_{it}; n_t, c_t) = b_{it} + \frac{G_{M|B}\left(b_{it}|b_{it}, n_t, c_t\right)}{g_{M|B}\left(b_{it}|b_{it}, n_t, c_t\right)}$$

(recall: LHS is i's "pivotal expectation" of the value of winning)

- if no unobserved heterogeneity...
  - RHS is observable
  - $\implies$ identification of $F_w\left(w(S_{1t}; n_t, c_t), \ldots, v(S_{nt}; n_t, c_t)\right))$
- with unobserved heterogeneity: we can't condition on all components of C_t (bidders condition on U_t, but we can't).

A Useful Fact

Separability is preserved by equilibrium bidding

(Haile-Hong-Shum (2003), Athey-Haile (2006), Krasnokutskaya (2011))

- suppose $V_{it} = V_{it}^{0}\,\Gamma(C_t)$ where $V_{it}^{0} \perp\!\!\!\perp C_t$

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- then

$$\begin{array}{ccc} \underbrace{\beta\left(S_{it}; n_t, x_t, u_t\right)} & = & \Gamma\left(x_t, u_t\right) \underbrace{\beta^0\left(S_{it}; n_t\right)} \\ B_{it} & = & B_{it}^0 \end{array}$$

where β^0 denotes the symmetric BNE bidding strategy for an auction t at which $\Gamma\left(X_t, U_t\right) = \Gamma\left(x^0, 0\right) \underset{\text{wlog}}{=} 1$.

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where β^0 denotes the symmetric BNE bidding strategy for an auction t at which $\Gamma\left(X_t, U_t\right) = \Gamma\left(x^0, 0\right) \underset{\text{wlog}}{=} 1$.

$\Rightarrow$ "homogenized" bid B_{it}^0, valuation V_{it}^0, expectation $w^0(S_{it}; N_t)$ (those i would have had in a "standardized" auction where $\Gamma(X_t, U_t) = 1$).

Control Function Approaches

Control Function Approach

Making the unobservable observed

- simple idea: if some other observable outcome (not bids) responds to U_t, condition on that outcome to indirectly condition on U_t.

Bidder Entry as a Control

Campo, Perrigne and Vuong (2003); Haile, Hong and Shum (2003); Guerre, Perrigne and Vuong (2009)

- suppose $N_t = \eta(X_t, Z_t, U_t)$, where $\eta(x, z, \cdot)$ is strictly increasing for all x, z (requires discrete U_t)
- then conditioning on (N_t, X_t, Z_t) implicitly conditions on U_t
- so a correct FOC is

$$w(s_{it}; n_t, x_t, u_t) = b_{it} + \frac{G_{M|B}(b_{it}|b_{it}, n_t, x_t, z_t)}{g_{M|B}(b_{it}|b_{it}, n_t, x_t, z_t)}.$$

- RHS observable $\implies$ identification of $F_w(\cdot|n_t, c_t)$.

Generalizations

- with an exclusion restriction on Z_t,
  - can identify the entry function η, each u_t (as a "residual"), F_U, and $F_w(\cdot|x, z, u)$
  - generate exogenous variation in N_t (sometimes useful for identification or model testing)
- other "control" outcomes (Athey and Haile (2006))
  - Roberts (2014) uses the seller's reserve price, which avoids the need for discrete U_t.

Measurement Error Approaches

Measurement Error Approach

Broad ideas:

- avoid the strict monotonicity requirement by using *multiple* outcomes that are only *stochastically* increasing in U_t
- such outcomes are like noisy measurements of U_t
- in the independent private values ("IPV") model, bids themselves can serve as these measures.

Kotlarski's Lemma

Two independent measurements of a latent variable X^*

- suppose
  - $X_1 = X^* + \eta_1$
    $X_2 = X^* + \eta_2$
  - $X^* \perp\!\!\!\perp \eta_1 \perp\!\!\!\perp \eta_2$
  - $E[\eta_1] = 0$, non-vanishing CF
- Kotlarski (1967) showed identification of the distribution $F(X^*, \eta_1, \eta_2) = F_X(X^*) F_{\eta_1}(\eta_1) F_{\eta_2}(\eta_2)$
- many applications: measurement error with repeated measurements, panel data etc.

Krasnokutskaya (2011)

IPV model with Multiplicative (or additive) UH

- suppose $V_{it} = U_t \ V_{it}^0$, with $U_t \perp\!\!\!\perp V_{1t} \perp\!\!\!\perp V_{1t} \perp\!\!\!\perp \ldots \perp\!\!\!\perp V_{N_t t}$
- preservation of separability $\implies$

$$B_{it} = U_t B_{it}^0$$

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- preservation of separability $\implies$

$$B_{it} = U_t B_{it}^0$$

- consider two bidders, wlog $i = 1, 2$

$$\ln B_{1t} = \ln U_t + \ln B_{1t}^0$$
$$\ln B_{2t} = \ln U_t + \ln B_{2t}^0$$

- apply Kotlarski $\implies$ distribution of (U_t, B_{1t}, B_{2t}) identified
- use the FOC to recover the joint distribution of (V_{it}, U_t)

Identification with Nonclassical Measurement Error

Hu (2008)

- suppose

$$Y \perp\!\!\!\perp X \perp\!\!\!\perp Z | X^*$$

  - Y: dependent variable
  - X^*: latent independent variable (discrete)
  - X: mismeasured indicator of X^* (discrete)
  - Z: instrument (discrete)

- Hu (2008): joint distribution of (Y, X, Z, X^*) identified under full rank condition + monotonicity condition.

Hu, McAdams and Shum (2013)

IPV model with UH again, but without separability:

- $V_{1t} \perp\!\!\!\perp V_{2t} \perp\!\!\!\perp ... \perp\!\!\!\perp V_{N_t t} | U_t$
- $U_t \in \{1, ..., J\}$ (discrete UH)

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Apply Hu (2008):

- consider 3 bidders, wlog $i = 1, 2, 3$
- discretize B_{2t}, B_{3t} as $\bar{B}_{2t}, \bar{B}_{3t}$
- then $B_{1t} \perp\!\!\!\perp \bar{B}_{2t} \perp\!\!\!\perp \bar{B}_{3t} | U_t$
- monotonicity assumption: $V_{it} | U_t$ FOSD-increasing in U_t
- by Hu (08), joint distn of (B_{it}, U_t) identified for each i
- now use the FOC to recover the marginals $F_{V_i}(\cdot | U)$.

Existing Approaches

Each has limitations, e.g.,

- strict monotonicity
- discrete UH
- separability
- limited to IPV models

Next: two new approaches—also with limitations, but may be better suited for some applications.

A "Quasi-Control Function" Approach

Compiani, Haile and Sant'Anna (2017)

Consider a relaxed version of the Haile, Hong and Shum (2003) entry model:

$$N_t = \eta(X_t, Z_t, U_t)$$

- $U_t | X_t, Z_t \sim u[0,1]$
- η is *weakly* increasing in U_t

Goal: relax the most problematic aspect of the HHS control function approach (strict monotonicity/discrete UH) while retaining its most attractive features (applicability beyond IPV, IV approach to isolating exogenous variation in N_t).

Exclusion and Separability

1. $F_{SV}(S_t, V_t | N_t, Z_t, X_t, U_t) = F_{SV}(S_t, V_t | N_t, X_t, U_t)$
2. $V_{it} = \Gamma(X_t, U_t) V_{it}^0$, where Γ bounded and wkly incr in U_t
3. Given $N_t = n$, $(V_{1t}^0, \ldots, V_{nt}^0, S_{1t}, \ldots, S_{nt}) \perp\!\!\!\perp (X_t, U_t)$.

Note:

- 1 imposes the *exclusion* half of "Z_t is IV for N_t"
- $2 + 3 \Longrightarrow$
  - (X_t, U_t) affect (S_t, V_t) only through separable index $\Gamma(X_t, U_t)$
  - weak monotonicity of valuations in U_t.

Example: Entry and Bidding in U.S. Oil Lease Auctions

Standard entry model (e.g., Berry 1992), followed by bidding

- tract t has A_t active *neighbor tracts*, owned by Z_t distinct *neighbor firms*
- firms (neighbor, other) must invest to obtain a signal and bid
- $E_t \in \mathbb{R}^K$ unobservable, possibly correlated with X_t, scales valuations through an unrestricted index $\psi(X_t, E_t)$
- two-stage game: entry then bidding
- perfect Bayesian eqm $\implies$ model above, where
  - U_t now a scalar unobservable, interpretation varying with X_t
  - Z_t a valid instrument.

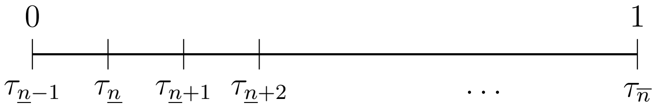
Identification in Three Steps

1. identification of the entry model
2. identification of the index function Γ
3. identification of joint distn of $(w(S_{1t}; n, x, u), \ldots, w(S_{nt}; n, x, u))$.

1. Identification of the Entry Model: Sketch

Bounds on each u_t from entry probabilities:

- $\eta(x,z,\cdot)$ is completely characterized by thresholds $\tau_n(x,z)$, $n = \underline{n}(x,z) - 1, \ldots, \overline{n}(x,z)$. For given (x,z),



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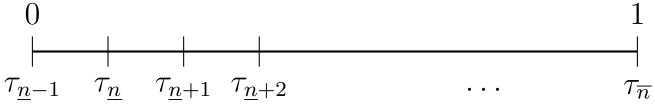

- we observe $\Pr(N_t = n | X_t = x, Z_t = z) = \tau_n(x, z) - \tau_{n-1}(x, z)$
- solve for all thresholds $\tau_n(x, z)$ (using endpoints 0 and 1)
- $\rightarrow$ bounds on each u_t: conditional on x, z, n, $U_t \sim u[\tau_{n-1}(x, z), \tau_n(x, z)]$

(Could now pursue partial id of model without the separabilty assumption. Instead we exploit the way that the bounds on U_t move with z).

2. Identification of the Index Function: Sketch

Exploit separability and shifting entry thresholds:

- $\sup \{ \ln B_{it} | N_t = n, X_t = x, Z_t = z \} = \gamma(x, \tau_n(x, z)) + \ln \beta^0(\bar{s}; n)$

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- difference again to obtain (some) differences of the form $\gamma(\hat{x}, \tau_{\hat{n}}(\hat{x}, \hat{z})) - \gamma(x, \tau_{n-1}(x, z))$ for $\hat{n} = n$ and $n - 1$

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- similar double differencing using inf instead of sup
- setting $\gamma(x^0, 0) = 0$ wlog, link differences to learn $\gamma(x, u)$ at all x, u.

Suppressed Assumptions

Loosely:

- U_t and Z_t are substitutes in the "production" of entry
- Z_t variation sufficient offset certain discrete variation in U_t — roughly, variation altering entry by 1 bidder

(no large support or completeness).

3. Identification of the Bidding Model: Sketch

Deconvolution + FOC

- $(\ln B_{1t}, \ldots, \ln B_{nt})$ observed
- fix $X_t = x, N_t = n$ and recall $\ln B_{it} = \ln B_{it}^0 + \gamma(x, U_t)$
- the random variable $\gamma(x, U_t)$
  - is independent of $\left(\ln B_{1t}^0, \ldots, \ln B_{nt}^0\right)$
  - has (now) known distribution

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(repeat at another x to obtain falsifiable restriction of model)

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  (repeat at another x to obtain falsifiable restriction of model)
- with γ, determines distn of $B_{it} + \frac{G_{M|B}(B_{it}|B_{it},n,x,u)}{g_{M|B}(B_{it}|B_{it},n,x,u)}\ \forall u$
- FOC$\rightarrow$joint distn of $(w(S_{1t}; n, x, u), \ldots, w(S_{nt}; n, x, u))$.

Main Application: Testing for Common Values

Haile, Hong and Shum (2003): Exogenous variation in N_t produces

- data on all U.S. offshore oil and gas lease auctions 1954–1983
- under private values: no effect on marginal distribution of $w\left(S_{it}; N_t\right)$
- under common values: FOSD ordering (winner's curse)

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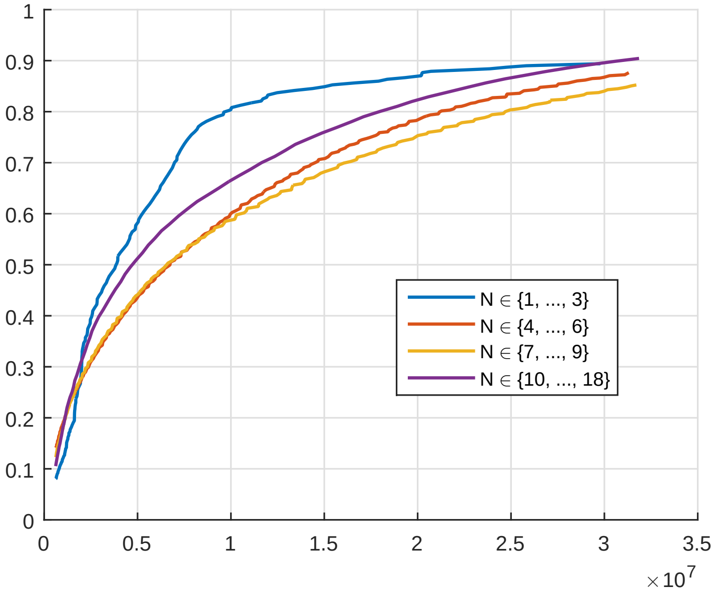
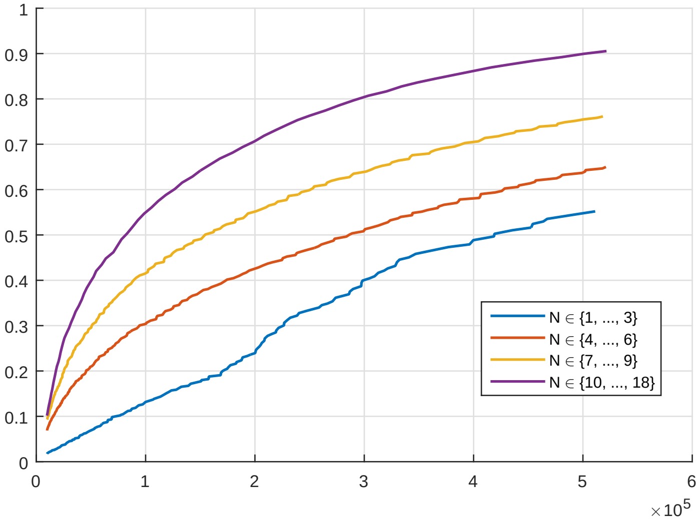
- estimation: two-step
  1. parametric/SNP estimation of entry thresholds $\tau(x, z, u)$ (ordered probit)
  2. semiparametric MLE
     - parametric index $\gamma(x, u)$
     - nonparametric marginal distn of homogenized bids (Bernstein)
     - parametric copula (gaussian).

Estimated Marginal CDFs, No UH



Estimated Marginal CDFs, with UH



A Finite Mixture Approach

Nonparametric Identification of Finite Mixtures

Kitamura and Laage (2016) consider nonparametric regression model

$$Y = \gamma(X, U) + \epsilon_U, \qquad \Pr\{U = j\} = \lambda_j, j = 1, ..., J$$

- key conditions for identification:
  - $\epsilon_U \perp\!\!\!\perp X$ for every $U \in \{1, ..., J\}$
  - $\exists$ segment (in x) where $\gamma(x, u), u \in \{1, ..., J\}$ are non-parallel
  - regularity condition (in terms of CF/MGF of $\epsilon_U, U \in 1, ..., J$)
- result: $\gamma(\cdot, u)$, distribution of ϵ_u, and λ_u nonparametrically identified for every $u \in \{1, ..., J\}$

An Affiliated Values Auction Model

Consider the following structure:

- $w\left(S_{it}; N_t, X_t, U_t\right) = \Gamma(N_t, X_t, U_t) w^0\left(S_{it}; N_t, U_t\right)$
- given $N_t = n$, $(S_{1t}, ...S_{nt}) \perp\!\!\!\perp X_t$
- support of U_t is $\{1, ..., J\}$, with $\Pr\{U_t = j\} := \lambda_j$

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Like before, $w^0(\cdot)$ is pivotal expected valuation after homogenization. But here it is allowed to depend on U_t: i.e., separability assumption here is only wrt X_t.

Model requires discrete UH but relaxes the separability requirement of the quasi-control function and measurement error approaches, while avoiding the restriction to IPV models required by ME approaches.

Transforming the Auction Model

- separability still preserved by equilibrium bidding:

$$\underbrace{\beta(S_{it}; n_t, x_t, u_t)}_{B_{it}} = \Gamma(x_t, u_t) \cdot \underbrace{\beta^0(S_{it}; n_t, u_t)}_{B_{it}^{0,u_t}}.$$

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- fix $N_t = n$ and take any $c = (c_1, ..., c_n)' \in \mathbf{R}^n$
- construct linear combination of log bids:

$$\underbrace{\sum_{i=1}^{n} c_i \ln B_{it}}_{\equiv \tilde{B}_t^c} = \underbrace{\left(\sum_{i=1}^{n} c_i\right)}_{\equiv C} \gamma(X_t, U_t) + \underbrace{\sum_{i=1}^{n} c_i \ln B_{it}^{0,U_t}}_{\equiv \tilde{B}_t^{c,0,U_t}}$$

i.e.

$$\tilde{B}_t^c = C\gamma(X_t, U_t) + \tilde{B}_t^{c,0,U_t} \quad \text{with } \Pr\{U_t = j\} = \lambda_j$$

Applying the K-L Result

$$\tilde{B}_t^c = C\gamma(X_t, U_t) + \tilde{B}_t^{c,0,U_t}$$

- Kitamura-Laage delivers, for every c and each $u \in \{1, ..., J\}$, identification of $C\gamma(\cdot, u), \lambda_u$, distribution of $\tilde{B}_t^{c,0,u}$
- C is known, so $\gamma(\cdot, u)$ identified
- $c \in \mathbf{R}^n$ was arbitrary, so (marginal) distribution of every linear combination $\tilde{B}_t^{c,0,u}$ of $(\ln B_{1t}^{0,u}, ..., \ln B_{1n}^{0,u})$ is identified
- so by Cramér-Wold, joint distn of $(B_{1t}^{0,u}, ..., B_{nt}^{0,u})$ is identified

Applying the K-L Result

$$\tilde{B}_t^c = C\gamma(X_t, U_t) + \tilde{B}_t^{c,0,U_t}$$

- Kitamura-Laage delivers, for every c and each $u \in \{1, ..., J\}$, identification of $C\gamma(\cdot, u), \lambda_u$, distribution of $\tilde{B}_t^{c,0,u}$
- C is known, so $\gamma(\cdot, u)$ identified
- $c \in \mathbf{R}^n$ was arbitrary, so (marginal) distribution of every linear combination $\tilde{B}_t^{c,0,u}$ of $(\ln B_{1t}^{0,u}, ..., \ln B_{1n}^{0,u})$ is identified
- so by Cramér-Wold, joint distn of $(B_{1t}^{0,u}, ..., B_{nt}^{0,u})$ is identified
- since $\gamma(\cdot, \cdot)$ is now known, this implies identification of the joint distribution of $(B_{1t}, ..., B_{nt})|n, x, u$
- FOC as usual $\rightarrow$distn of $(w(S_{1t}; n, x, u), ..., w(S_{nt}, ; n, x, u))$.

Concluding Remarks

Auctions often offer applications where we have especially good knowledge of the "game" being played.

This allows econometricians to focus on exploration of particularly rich models with asymmetric information, asymmetries, risk aversion, etc. The resulting substantive/methodological insights often extend to other kinds of markets.

Today: unobserved "market level" heterogeneity. Several approaches to nonparametric identification, using insights from/for broader classes of models

- exact measure of the UH (control function)
- multiple noisy measures
- finite mixture models
- quasi-control function. (?)