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Technical Appendix to

HARNESSING WINDFALL REVENUES: OPTIMAL POLICIES FOR RESOURCE-RICH DEVELOPING ECONOMIES

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Appendix B. Saddlepath Dynamics (Section 3)

We establish three Propositions, which are used in the discussion of Section 3.1.

Proposition 1. *With a temporary 'small' windfall of size $N > 0$ from time $T_0 \geq 0$ to $T_1 > T_0$,*

$$\Delta C(0) + \Delta G(0) = \exp(-\lambda_u T_0)[1 - \exp(-\lambda_u T)]N > 0, \tag{B.1}$$

where

$$\lambda_u = \frac{1}{2}r^* + \frac{1}{2}\sqrt{r^{*2} + 8\sigma\Pi' Y} > r^* > 0,$$

$T \equiv T_1 - T_0 > 0$, *and Δ indicates deviations from the benchmark trajectories.*

Proof. Linearising (4) and (5) around a steady state with zero N, that is $F_\infty = \bar{F} = 0$, $\Pi_\infty = 0$ and $C_\infty = Y/(1+\psi^\sigma)$, yields:[1]

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \begin{pmatrix} \Delta N \\ 0 \end{pmatrix} \quad \text{with } \mathbf{x} \equiv \begin{pmatrix} \Delta F \\ \Delta C \end{pmatrix}, \mathbf{A} \equiv \begin{pmatrix} r^* & 1+\psi^\sigma \\ \Sigma & 0 \end{pmatrix} \quad \text{and} \quad \Sigma \equiv 2\sigma\Pi' C_\infty > 0.$$

The eigenvalues of the matrix $\mathbf{A}$ are:

$$\lambda_s = \frac{1}{2}r^* - \frac{1}{2}\sqrt{r^{*2} + 4\Sigma(1+\psi^\sigma)} < 0 \quad \text{and} \quad \lambda_u = \frac{1}{2}r^* + \frac{1}{2}\sqrt{r^{*2} + 4\Sigma(1+\psi^\sigma)} > r^* > 0.$$

As one of the eigenvalues is positive and the other one is negative, the system displays saddlepath dynamics. We solve this system with spectral decomposition of the matrix $\mathbf{A}$ (cf., Buiter, 1984):

$$\mathbf{A} \equiv \begin{pmatrix} r^* & 1+\psi^\sigma \\ \Sigma & 0 \end{pmatrix} = \mathbf{N}^{-1}\begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_u \end{pmatrix}\mathbf{N} \quad \text{with } \mathbf{N} \equiv \begin{pmatrix} N_{ss} & N_{su} \\ N_{us} & N_{uu} \end{pmatrix},$$

where the columns of the matrix $\mathbf{N}$ stack the eigenvectors of the matrix $\mathbf{A}$. The eigenvectors are calculated from the equations:

$$\mathbf{NA} = \begin{bmatrix} r^* N_{ss} + \Sigma N_{su} & (1+\psi^\sigma)N_{ss} \\ r^* N_{us} + \Sigma N_{uu} & (1+\psi^\sigma)N_{us} \end{bmatrix} = \begin{pmatrix} \lambda_s N_{ss} & \lambda_s N_{su} \\ \lambda_u N_{us} & \lambda_u N_{uu} \end{pmatrix} = \begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_u \end{pmatrix}\mathbf{N}.$$

[1] We assume $\Pi' > 0$ at the steady state. That is not strictly necessary as long as $\Pi' > 0$ at all points on the adjustment path towards the steady state.

Normalising such that $N_{uu} = 1$, we obtain $N_{us} = \lambda_u/(1+\psi^\sigma) = \Sigma/(\lambda_u - r^*) > 0$ from equating the two elements in the bottom row of the above matrix equation. Note that the top row gives $N_{ss} = -\Sigma/(1+\psi^\sigma) < 0$. Defining the vector $\mathbf{z} \equiv \mathbf{Nx}$, we obtain $z_u(t) = \int_0^\infty \exp(-\lambda_u t) N_{us} \Delta N(t) \mathrm{d}t$ provided we assume that $\lim_{t\to\infty} \exp(-\lambda_u t) z_u(t) = 0$ holds. Restricting the solution to the stable manifold, we obtain $\Delta C(t) = -N_{uu}^{-1} N_{us} \Delta F(t) + N_{uu}^{-1} z_u(t)$ or

$$\Delta C(t) = -N_{us}\Delta F(t) + z_u(t) = \left(\frac{\lambda_u}{1+\psi^\sigma}\right)\left\{\int_t^\infty \exp[-\lambda_u(t'-t)]\Delta N(t')\mathrm{d}t' - \delta F(t)\right\}.$$

With the step function $\Delta N(t) = N$, $T_0 < t \leq T_1$ and zero at all other instants of time, this equation becomes:

$$\begin{aligned}\Delta C(t) &= \left(\frac{\lambda_u}{1+\psi^\sigma}\right)\left\{\exp[-\lambda_u(T_0 - t)]\left[\frac{1-\exp(-\lambda_u T)}{\lambda_u}\right]N - \Delta F(t)\right\}, \quad 0 < t \leq T_0 \\ \Delta C(t) &= \left(\frac{\lambda_u}{1+\psi^\sigma}\right)\left(\left\{\frac{1-\exp[-\lambda_u(T_1-t)]}{\lambda_u}\right\}N - \Delta F(t)\right), \quad T_0 < t < T_1, \\ \Delta C(t) &= -\left(\frac{\lambda_u}{1+\psi^\sigma}\right)\Delta F(t), \quad t \geq T_1.\end{aligned} \tag{B.2}$$

As $\Delta F(0)$=0, it follows that

$$\Delta C(0) = \exp(-\lambda_u T_0)\left[\frac{1-\exp(-\lambda_u T)}{1+\psi^\sigma}\right]N > 0.$$

The initial jump in total consumption thus equals (B.1).

Proposition 2. *With a temporary 'small' windfall of size N starting at time* 0 *and finishing at* $t = T_1$, *we have* $\Delta F(t) < 0$ *for all* $t > 0$, $\Delta\dot{F}(t) < 0$ *for* $0 < t < T_1$ *and* $\Delta\dot{F}(t) > 0$ *for* $t > T_1$. *We also have* (7) *and* (8). *Comparing the outcome in the capital scarce economy with that under the PIH, we have* (10).

Proof. Setting $T_0 = 0$ and substituting the last two expressions for $\Delta C(t)$ given in (B.2) into $\Delta\dot{F}(t) = r^*\Delta F(t) + (1+\psi^\sigma)\Delta C(t) - \Delta N(t)$, and making use of $r^* - \lambda_u = \lambda_s$ yields:

$$\begin{aligned}\Delta\dot{F}(t) &= -\exp[-\lambda_u(T_1-t)]N + \lambda_s\Delta F(t), \quad 0 < t < T_1, \\ \Delta\dot{F}(t) &= \lambda_s\Delta F(t), \quad t \geq T_1.\end{aligned}$$

Solving these differential equations with the initial condition $\Delta F(0) = 0$, we obtain:

$$\begin{aligned}\Delta F(t) &= -\left[\frac{\exp(\lambda_u t) - \exp(\lambda_s t)}{\lambda_u - \lambda_s}\right]\exp(-\lambda_u T_1)N < 0, \quad 0 < t \leq T_1, \\ \Delta F(t) &= \exp[\lambda_s(t - T_1)]\Delta F(T_1) \to 0 \quad \text{as } t \to \infty, \quad t > T_1,\end{aligned} \tag{B.3}$$

where $\Delta F(T_1)$ is as given in (7). It follows that $\Delta F(t) < 0$ for all $t > 0$. Differentiation of (B.3) yields:

$$\begin{aligned}\Delta\dot{F}(t) &= -\left[\frac{\lambda_u\exp(\lambda_u t) - \lambda_s\exp(\lambda_s t)}{\lambda_u - \lambda_s}\right]\exp(-\lambda_u T_1)N < 0, \quad 0 < t \leq T_1, \\ \Delta\dot{F}(t) &= \lambda_s\exp[\lambda_s(t-T_1)]\Delta F(\mathrm{T}_1) > 0, \quad t > T_1,\end{aligned} \tag{B.4}$$

hence, $\Delta\dot{F}(t) < 0$, $0 < t < T_1$ and $\Delta\dot{F}(t) > 0$, $t > T_1$. Using (B.2), $\Delta C(T_1) + \Delta G(T_1) = -\lambda_u\Delta F(T_1)$ and thus using (7):

$$\Delta C(T_1) + \Delta G(T_1) = \left(\frac{\lambda_u}{\lambda_u - \lambda_s}\right)\{1 - \exp[(\lambda_s - \lambda_u)T_1]\}N > 0. \tag{B.5}$$

Hence, (8) follows. To establish the inequality sign in (10), we note that equation (B.5) becomes equal to the outcome under the PIH, $[1-\exp(-r^*T_1)]N$, if $\Sigma^* \equiv 4\Sigma(1+\psi^\sigma) = 0$. If we differentiate and use $\lambda_u + \lambda_s = r^*$, we find that:

$$\frac{\partial[\Delta C(T_1)+\Delta G(T_1)]}{\partial\Sigma^*}=\left[\frac{\exp[(\lambda_s-\lambda_u)T_1]N}{4(\lambda_u-\lambda_s)^3}\right]\langle -r^*\{1-\exp[(\lambda_s-\lambda_u)T_1]\}+2T_1\lambda_u(\lambda_u-\lambda_s)\rangle.$$

To sign the expression in the angular brackets, we note that it equals zero and its derivative with respect to T_1 is positive, $(2\lambda_u - r^*)(\lambda_u - \lambda_s) > 0$, if $T_1 = 0$. For positive T_1, this derivative is even more positive. Hence, the term in curly brackets is positive for all $T_1 > 0$ and therefore consumption at $t = T_1$ is more than under the PIH as indicated in expression (10).

Proposition 3. *With an anticipated 'small' windfall of size $N > 0$ starting at $t = T_0$ and finishing at $t = T_1$, we have*

$$\Delta\dot{F}(t) > 0 \quad \text{for } 0 < t < T_0, \quad \Delta\dot{F}(t) < 0 \quad \text{for } T_0 < t < T_1 \quad \text{and} \quad \Delta\dot{F}(t) < 0 \quad \text{for } t > T_1.$$

A bigger T_0 implies a bigger reduction in foreign debt at the end of the windfall.

Proof. Upon substitution of (B.2) into the differential equation for $\Delta F(t)$, we obtain:

$$\Delta\dot{F}(t) = \exp[-\lambda_u(T_0 - t)][1-\exp(-\lambda_u T)]N + \lambda_s\Delta F(t), \quad 0 < t \le T_0,$$
$$\Delta\dot{F}(t) = \{1-\exp[-\lambda_u(T_1 - t)]\}N + \lambda_s\Delta F(t), \quad T_0 < t < T_1,$$
$$\Delta\dot{F}(t) = \lambda_s\Delta F(t), \quad t \ge T_1.$$

Forward integration of this differential equation yields:

$$\Delta F(t) = \exp(\lambda_s t)[1-\exp(-r^* t)]\exp(-\lambda_u T)[1-\exp(-\lambda_u T)]N/r^* > 0, \quad 0 < t \le T_0,$$
$$\Delta F(t) = \left[\frac{\exp(\lambda_s t)-1}{\lambda_s}\right]\exp(-\lambda_u T_0)[1-\exp(-\lambda_u T)]N + \left\{\frac{1-\exp[\lambda_s(t-\mathrm{T}_0)]}{\lambda_s}\right\}N, \quad T_0 < t \le T_1,$$
$$\Delta F(t) = \exp[\lambda_s(t-T_1)]\Delta F(T_1) \to 0 \quad \text{as } t \to \infty, \quad t > T_1.$$

Ahead of the windfall ($t \le T_0$), the country borrows to make possible an increase in consumption $[\Delta F(t) > 0]$. During the windfall ($T_0 < t \le T_1$), the country starts to pay back its debt and eventually build up assets. At the end of an anticipated windfall, we have:

$$-\Delta F(T_1) = -\left(\frac{N}{\lambda_s}\right)\{1-\exp(\lambda_s T_1) - [1-\exp(\lambda_s T_1)]\exp(-\lambda_u T_0)[1-\exp(-\lambda_u T)]\} > 0.$$

Note that

$$\frac{\partial - \Delta F(T_1)}{\partial T_0} = -\left(\frac{N}{\lambda_s}\right)(-\lambda_s\exp(\lambda_s T_1) + \{\lambda_u\exp(-\lambda_u T_0) - (\lambda_u-\lambda_s)\exp[-(\lambda_u-\lambda_s)T_0 + \lambda_s T]\}$$
$$\times[1-\exp(-\lambda_u T)])$$

and thus

$$\frac{\partial - \Delta F(T_1)}{\partial \mathrm{T}_0} = -\left(\frac{N}{\lambda_s}\right)\{-\lambda_s\exp(\lambda_s T)+[\lambda_u - (\lambda_u-\lambda_s)\exp(\lambda_s T)][1-\exp(-\lambda_u T)]\} > 0$$

at $T_0 = 0$. Hence, the bigger T_0, the more debt is reduced at the end of the windfall. After the windfall ($t > T_1$), the second term in the expression for $\Delta F(t)$ gradually vanishes as $t \to \infty$.

Appendix C. Comparative Statics of Production

Section 4: The production function is $Y = f(K, S)$ (with labour force normalised at unity and constant return to private capital and labour). Profit maximisation implies $(1 - \tau)\, f_K(K, S) = r^* + \delta_K$ this implicitly defining $K(S, \tau)$. Totally differentiating, $f_{KK}\mathrm{d}K = (f_K \mathrm{d}\tau)/(1-\tau) - f_{KS}\mathrm{d}S$. The wage is $W(S,\tau) = (1-\tau)\{f[K(S,\tau),S] - K(S,\tau)f_K[K(S,\tau),S]\}$ so that

$$\mathrm{d}W = W_\tau\, \mathrm{d}\tau + W_S\, \mathrm{d}S = -f[K(S, \tau), S]\mathrm{d}\tau + (1 - \tau)f_S[K(S, \tau), S]\mathrm{d}S$$

(so $W_\tau = -Y$)

Income is $Y(S, \tau) = f[K(S, \tau), S]$. Totally differentiating gives

$$\mathrm{d}Y = Y_\tau \mathrm{d}\tau + Y_S \mathrm{d}S = \frac{(f_K)^2}{f_{KK}}\left(\frac{\mathrm{d}\tau}{1-\tau}\right) + \left(f_S - \frac{f_K f_{SK}}{f_{KK}}\right)\mathrm{d}S.$$

For the Cobb–Douglas case $Y = L^{1-\alpha}K^{\alpha}S^{\gamma}$ (with $L = 1$), we have

$$Y_\tau = \frac{(f_K)^2}{f_{KK}}\left(\frac{1}{1-\tau}\right) = \left(\frac{\alpha}{\alpha-1}\right)\left(\frac{1}{1-\tau}\right), \quad Y_S = \left(f_S - \frac{f_K f_{SK}}{f_{KK}}\right) = \left(\frac{\gamma}{1-\alpha}\right)\left(\frac{Y}{S}\right)$$

and $W_S = (1 - \alpha)(1 - \tau)Y_S$.

Section 5: Profit maximisation implies $f_K(K) - \delta_K = r = r^* + \Pi(K - E)$, this implicitly defining $K(E)$. Hence, $r(E) = f_K[K(E)] - \delta_K$ and $W(E) = f[K(E)] - K(E)f_K[K(E)]$. Comparative statics are;

$$K_E = \Pi'/(\Pi' - f_{KK}), r_E = \Pi' f_{KK}/(\Pi' - f_{KK}), W_E = -\Pi' K f_{KK}/(\Pi' - f_{KK}), \quad \text{so } W_E + K r_E = 0.$$