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Technical Appendix to

FISCAL POLICY IN A TRACTABLE LIQUIDITY-CONSTRAINED ECONOMY

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Appendix A. Liquidity-constrained Workers

A.1. *Proof of Proposition 1*

If fluctuations around the steady state are sufficiently small, then C1 and C2hold in every period provided that they hold in the steady state. The condition $B + \beta\delta > 0$ implies that either $B > 0$ or $\delta > 0$ or both, and hence that $a^e > 0$ from equation (23); thus, conjecture C2 holds provided that $B + \beta\delta > 0$. What is left to establish is that conjecture C1 holds in the steady state provided that conditions (i)–(iv) in the proposition are satisfied. We proceed in two steps. First we show that $\beta R < 1$ if and only if $B + \beta\delta < \Sigma$ (step 1). Second, we show that C1 holds whenever $\beta R < 1$ and conditions (i)–(iii) are satisfied (step 2).

Step 1. To prove that $R\beta < 1$ if and only if $B < \Sigma - \beta\delta$, we show that $B(R)$ is a continuous, strictly increasing function of R over the appropriate interval and that $B(1/\beta) = \Sigma - \beta\delta$ [so that $B(R) < \Sigma - \beta\delta \Leftrightarrow R < 1/\beta$]. First, let us rewrite the steady-state counterpart of (17) as follows:

$$u'(c^{eu}) = [(\beta R)^{-1} - \pi^e]/(1 - \pi^e), \tag{A.1}$$

By assumption $G = 0$, implying that $T = B(R - 1)$. Thus, after some manipulations the steady-state counterpart of (26) can be written as:

$$B = \tilde{B}(R) - \delta/R \equiv B(R), \tag{A.2}$$

where

$$\mathring{B}(R) \equiv \left[\frac{1-\Omega}{1+\Omega(R-1)}\right] u'^{-1}\left[\frac{(\beta R)^{-1} - \pi^e}{1-\pi^e}\right].$$

The term $-\delta/R$ in (A.2) is strictly increasing in R when $\delta > 0$. Moreover, we have:

$$\mathring{B}'(R) = \frac{-(1-\Omega)\Omega}{[1+\Omega(R-1)]^2} u'^{-1}\left[\frac{(\beta R)^{-1} - \pi^e}{1-\pi^e}\right] + \left[\frac{1-\Omega}{1+\Omega(R-1)}\right]\frac{\partial}{\partial R} u'^{-1}\left[\frac{(\beta R)^{-1} - \pi^e}{1-\pi^e}\right]. \tag{A.3}$$

Equation (A.1) implies that $u'[c^{eu}(R)] = [(\beta R)^{-1} - \pi^e]/(1-\pi^e)$, so the $\partial u'^{-1}(.)/\partial R$ term above is:

$$\frac{\partial}{\partial R} u'^{-1}\left[\frac{(\beta R)^{-1} - \pi^e}{1-\pi^e}\right] = \frac{1}{u''(c^{eu})} \times \frac{\partial u'(c^{eu})}{\partial R} = \frac{1}{u''(c^{eu})} \times \frac{-1}{(1-\pi^e)\beta R^2}.$$

After rearranging, this allows us to rewrite (A.3) as follows:

$$\begin{aligned}\mathring{B}'(R) &= \frac{-(1-\Omega)\Omega c^{eu}}{[1+\Omega(R-1)]^2} + \left[\frac{1-\Omega}{1+\Omega(R-1)}\right] \times \frac{-R^{-2}}{(1-\pi^e)\beta} \times \frac{1}{u''(c^{eu})} \\ &= \frac{(1-\Omega)\Omega u'(c^{eu})}{[1+\Omega(R-1)]^2 u''(c^u)} \left\{-\frac{c^{eu}u''(c^{eu})}{u'(c^{eu})} - \frac{[1+\Omega(R-1)]}{\Omega R^2(1-\pi^e)\beta u'(c^{eu})}\right\} \\ &= \frac{(1-\Omega)\Omega u'(c^{eu})}{[1+\Omega(R-1)]^2 u''(c^{eu})} \left[\sigma(c^{eu}) - \frac{1+\Omega(R-1)}{(1-\pi^e\beta R)\Omega R}\right].\end{aligned}$$

The term inside the pair of large brackets must be negative for $\mathring{B}'(R)$ to be positive. Since $\sigma(c) \leq 1$ by assumption, a sufficient condition for this is $[1+\Omega(R-1)]/(1-\pi^e\beta R)\Omega R > 1$, which is always true. Thus, $\mathring{B}(R)$ is continuous and strictly increasing in over $(0, 1/\beta\pi^e)$ and, from the definition of $\mathring{B}(R)$, we have the boundaries:

$$\lim_{R\to 0} \mathring{B}(R) = u'^{-1}(\infty)(=0), \quad \lim_{R\to 1/\beta\pi^e} \mathring{B}(R) = \frac{\beta\pi^e(1-\Omega)u'^{-1}(0)}{1+\Omega(1-\beta\pi^e)}(\leq\infty).$$

This in turn implies that $B(R)$ in (A.2) is continuous and strictly increasing in R.

When $\delta = 0$, we have that $B(R) = \mathring{B}(R)$, and the inequality $B(R) < \Sigma$ is recovered by evaluating $\mathring{B}(R)$ at $R = 1/\beta$. When $\delta > 0$, the maximum possible value of R is still $1/\beta\pi^e$, with $\lim_{R\to 1/\beta\pi^e} B(R) = \lim_{R\to 1/\beta\pi^e} \mathring{B}(R)$. The lowest possible value of R, denoted $R_{\min}$, corresponds to the point at which $B(R_{\min}) = 0$. Hence, from (A.2), $R_{\min}$ is the (unique) solution to $\mathring{B}(R) = \delta/R$, and by construction we have that $\lim_{R\to R_{\min}} B(R) = 0$. Again, the equivalence between $\beta R < 1$ and $B + \beta\delta < \Sigma$ follows from the increasingness of the $B(R)$ function and its evaluation at $R = 1/\beta$. (Note also that $R_{\min} < 1/\beta$ since $\mathring{B}(1/\beta) > \delta\beta$ under condition *iv* in the proposition.)

Step 2. We must now show that $\beta R < 1$ is a sufficient condition for conjecture C1 to hold when conditions (*i*)–(*iii*) in the proposition also hold. For C1 to hold, both *eu* and *uu* workers must be borrowing-constrained in the steady state, so that we must have:

$$u'(c^{eu}) > \beta R(1-\pi^u)u'(c^e) + \beta R\pi^u u'(c^{uu}), \tag{A.4}$$

$$u'(c^{uu}) > \beta R(1-\pi^u)u'(c^e) + \beta R\pi^u u'(c^{uu}), \tag{A.5}$$

with $u'(c^e) = 1$ [see equation (16)]. The right-hand side of (A.4) and (A.5) are the expected marginal utility of future consumption for an unemployed worker, which in our conjectured equilibrium is the same whether the worker is *eu* or *uu*. Hence, there are two cases to consider. If $c^{uu} \geq c^{eu}$, then $u'(c^{uu}) \leq u'(c^{eu})$ and (A.5) is a sufficient for (A.4)–(A.5) to hold; on the contrary, if $c^{eu} > c^{uu}$, then $u'(c^{eu}) < u'(c^{eu})$ and (A.4) is a sufficient condition for (A.4)–(A.5) to hold.

Case 1. Assume that $c^{uu} \geq c^{eu}$, so that (A.5) is the relevant sufficient condition. The inequality holds for $\pi^u \to 1$ whenever $\beta R < 1$. Then, since $u'(c^{uu}) > u'(c^e)$ (because $c^{uu} < c^e$, otherwise the employed would be constrained), it follows that the inequality holds for all $\pi^u \in [0, 1)$.

Case 2. Assume that $c^{eu} > c^{uu}$, so that (A.4) is the relevant sufficient condition. Using (A.1), we may rewrite (A.4) as follows:

$$\frac{1-\beta\pi^e R - (1-\pi^e)\beta^2R^2}{(1-\pi^e)\beta^2R^2} > \pi^u[u'(c^{uu}) - u'(c^e)].$$

The fact that $\beta R < 1$ ensures that the left hand side of this inequality is positive, while $c^{uu} < c^e$ implies the right hand side also is. Thus, the inequality holds provided that π^u is sufficiently small [condition (*ii*) in Proposition 1].

Maximum public debt–output ratio. Note that the condition according to which $B < \Sigma - \delta\beta$ can be expressed equivalently as a condition on the maximum level of the steady-state public debt–output ratio, B/Y, consistent with the liquidity-constrained equilibrium. Using (28), (22), the fact that $T = B(R-1)$ and the definitions of Ψ and Υ in Section 2.3, we find that the output–public debt ratio can be expressed as follows:

$$\frac{Y}{B} = \Omega + \frac{1}{B}\left(\Psi + \frac{\Omega\delta}{R}\right) + \Upsilon R.$$

Recall that R increases with B, so the second term in the right-hand side of the latter equation falls with B while the third term rises with B. However, as π^u becomes small [condition (*ii*) in the proposition], ΥR becomes small relative to $(\Psi + \Omega\delta/R)B^{-1}$, thus the second term determines how B affects Y/B. This implies that the public debt–output ratio B/Y rises with B, so that we can find the maximum value of B/Y consistent with our limited-heterogeneity equilibrium by evaluating B/Y at $R = 1/\beta$. After some rearrangements, we obtain:

$$\left(\frac{B}{Y}\right)_{R=1/\beta} = \left(\frac{\Omega\Sigma + \Psi}{\Sigma - \delta\beta} + \frac{\Upsilon}{\beta}\right)^{-1}.$$

A.2. *Dynamics and stability*

We use hatted variables to denote level-deviations from steady state (i.e. $\hat{X}_t = X_t - X$). First, substitute (9)–(10) into the linearised versions of (8) and (26) to obtain:

$$\hat{B}_t = \left(\frac{B}{1+\phi}\right)\hat{R}_{t-1} + \left(\frac{R}{1+\phi}\right)\hat{B}_{t-1} + \left(\frac{1}{1+\phi}\right)G_t + \left(\frac{1}{1+\phi}\right)T_t^c, \tag{A.6}$$

$$\hat{R}_t = \frac{-R^2\beta(1-\pi^e)u''(c^{eu})}{1-\Omega}\left[(1+\Omega\phi)E_t(\hat{B}_{t+1}) - \psi G_t - \Omega\chi T_t^c\right] - (1-\pi^e)\beta u''(c^{eu})\delta E_t(\hat{R}_{t+1}), \tag{A.7}$$

where, from (A1) to (A2),

$$c^{eu} = u'^{-1}\left[\frac{(\beta R)^{-1} - \pi^e)}{1-\pi^e}\right] = \frac{(B + \delta/R)[1 + \Omega(R-1)]}{1-\Omega}.$$

Equations (A.6)–(A.7) define a two-dimensional backward/forward dynamic system, with sequences of unknowns $\{\hat{B}_t\}_{t=0}^{\infty}$ and $\{\hat{R}_t\}_{t=0}^{\infty}$ and forcing sequences $\{G_t\}_{t=0}^{\infty}$ and $\{T_t^c\}_{t=0}^{\infty}$. The solution to this system takes the form of a VAR whose coefficients can be recovered by the method of undetermined coefficients. More specifically, let $X_t \equiv [\,\hat{B}_t\ \hat{R}_t\,]'$ and $Z_t \equiv [\,G_t\ T_t^c\,]'$. Leading (A.6) one period, taking expectations and substituting (10) and (A.7) into the resulting equation, we can express the dynamics of the model in matrix form as $X_t = ME_t(X_{t+1}) + NZ_t$, where M and N are conformable matrices whose coefficients are functions of the deep parameters of the model. There are two cases to consider, depending on whether $\delta = 0$ or $\delta > 0$.

Case 1. When $\delta = 0$, M is singular, implying that the solution dynamics of the model is univariate. To see why this is the case, let us rewrite (A.7) as follows, making use of (A.1)–(A.2) and the definition of ρ in (33):

$$(\rho/B)E_t\left[(1+\Omega\phi)\hat{B}_{t+1} - \psi G_t - \Omega\chi T_t^c\right] = \hat{R}_t. \tag{A.8}$$

Leading (A.3) one period and taking expectations, solving (A.5) for $E_t(\hat{B}_{t+1})$, and then equating the two expressions, we obtain:

$$R\hat{B}_t + \psi\left[1 - \frac{(1+\phi)}{1+\Omega\phi}\right]G_t + \chi\left[1 - \frac{(1+\phi)\Omega}{(1+\Omega\phi)}\right]T_t^c = B\left[\frac{(1+\phi)}{\rho(1+\Omega\phi)} - 1\right]\hat{R}_t.$$

Now, lagging the latter equation one period, solving it for $\hat{R}_{t-1}$ and substituting the resulting expression into (A.3), one finds equation (32) with coefficients:

$$\gamma = \frac{R}{1+\phi-\rho(1+\phi\Omega)}, \quad \mu = \frac{1}{1+\phi}, \quad \nu = -\frac{\gamma\rho\psi\phi(1-\Omega)}{R(1+\phi)}, \quad \upsilon = \frac{\gamma\rho\chi(1-\Omega)}{R(1+\phi)},$$

where R is uniquely defined by B (see Appendix A.1).

The sign of γ is related to the stationarity requirement that $|\gamma| < 1$. In the case where $\gamma > 0$, then a necessary and sufficient condition for stationarity is (33) in the body of the paper, given that $1 - \rho\Omega > 0$. Does a stationary path for B_t exist consistent with the case where $\gamma < 0$? If $\gamma < 0$, the necessary and sufficient condition for stationarity becomes $\phi < (-1 - R + \rho)/(1 - \Omega\rho)$, but the right-hand side of this inequality is negative. Since this is inconsistent with $\phi > 0$, it must be the case that (33) holds, which in turn implies that $\gamma > 0$. By implication $\nu < 0$, and obviously $\mu > 0$ and $\upsilon > 0$ since $\phi > 0$. Finally, with $\gamma > 0$ and $1 - \rho\Omega > 0$ we have $\partial\gamma/\partial\phi < 0$.

Case 2. When $\delta > 0$, M is invertible and the solution dynamics are bivariate. First, rewrite the forward-looking dynamics $X_t = ME_t(X_{t+1}) + NZ_t$ as follows:

$$E_t(X_{t+1}) = M^{-1}X_t - M^{-1}NZ_t. \qquad (A.9)$$

We know from the literature on expectational linear systems (e.g. Uhlig, 2001) that the solution to (A.9) has the following VAR representation:

$$X_t = \tilde{M}X_{t-1} + \tilde{N}Z_t, \qquad (A.10)$$

where $\tilde{M}$ and $\tilde{N}$ are matrices to be determined. Leading (A.10) one period, taking expectations and using (A.9) enables us to fully identify $\tilde{M}$and $\tilde{N}$. We may then verify numerically that for the parameter configurations considered when running impulse-response functions ϕ is sufficiently large for $\{X_t\}_{t=0}^{\infty}$ to remain stationary.

A.3. *Imperfectly elastic labour supply*

For simplicity we assume here that $\delta = 0$, but nothing peculiar hinges on this assumption. With imperfectly inelastic labour supply, asset accumulation is gradual, and not all working agents end the period with the same asset wealth. We denote by a_t^{ue} and a_t^{ee} the end-of-period asset wealth of *ue* and *ee* workers, respectively, and by c_t^{ue} and that c_t^{ee} the corresponding individual consumption levels. By assumption, all *ee* workers pool their asset wealth at the beginning of the period, so any *ee* worker turns out starting the period with individual asset $(\omega^{ee}a_{t-1}^{ee} + \omega^{ue}a_{t-1}^{ue})/(\omega^{ee} + \omega^{ue})$ – there are $\pi^e\omega^{ee}$ and $\pi^e\omega^{ue}$ workers entering date t with wealth a_{t-1}^{ee} and a_{t-1}^{ue}, respectively, and the total number of *ee* workers is $\omega^{ee} = \pi^e(\omega^{ee} + \omega^{ue})$.

Since $\omega^{ee}/(\omega^{ee} + \omega^{ue}) = \pi^e$ and $\omega^{ue}/(\omega^{ee} + \omega^{ue}) = 1 - \pi^e$, we may rewrite the budget constraint of an *ee* worker as follows:

$$ee : c_t^{ee} + a_t^{ee} = [\pi^e a_{t-1}^{ee} + (1-\pi^e)a_{t-1}^{ue}]R_{t-1} + l_t^{ee} - T_t. \qquad (A.11)$$

Since *eu* workers hold end-of-period wealth level a_t^{ue}, their budget constraint is:

$$ue : c_t^{ue} + a_t^{ue} = l_t^{ue} - T_t. \qquad (A.12)$$

Note that workers who fall into unemployment at date t can now be of two different types, depending on their asset holdings at the end of date $t-1$ with (i.e. a_{t-1}^{ee} or a_{t-1}^{ue}), which in turn depends on their labour statuses at dates $t-1$ and $t-2$. These two types (i.e. '*eeu* workers' and '*ueu* workers') have the following budget constraints:

$$eeu : c_t^{eeu} = a_{t-1}^{ee} R_{t-1} - T_t, \quad ueu : c_t^{ueu} = a_{t-1}^{ue} R_{t-1} - T_t.$$

Finally, the budget constraint of *uu* workers is unchanged (i.e. $uu : c_t^{uu} = \kappa - T_t$).

The optimal asset demand and labour supply decisions of employed workers are as follows: *ee* workers, who end the current period with wealth a_t^{ee}, remain employed with π^e, in which case they remain of the *ee* type, or fall into unemployment, in which case they become of the *eeu* type. Similarly, *ue* workers stay employed and become *ee* or fall into unemployment and become *ueu*. Hence the optimal asset demand and labour supply decisions of *ee* and *ue* workers must satisfy:

$$ee : u'(c_t^{ee}) = \beta\pi^e R_t u'(c_{t+1}^{ee}) + \beta(1-\pi^e) R_t E_t u'(c_{t+1}^{eeu}), \quad u'(c_t^{ee}) = v'(l_t^{ee}),$$
$$ue : u'(c_t^{ue}) = \beta\pi^e R_t u'(c_{t+1}^{ee}) + \beta(1-\pi^e) R_t E_t u'(c_{t+1}^{ueu}), \quad u'(c_t^{ue}) = v'(l_t^{ue}).$$

Finally, the bond market clearing condition must be modified to account for the fact that $a_t^{ee} \neq a_t^{ue}$ whenever $v'(l) \neq 1$. It is now given by $\omega^{ee} a_t^{ee} + \omega^{ue} a_t^{ue} = B_t$.

Note that when labour supply becomes perfectly elastic [i.e. $v'(l) = 1\ \forall l$, as in our baseline utility function], the intratemporal optimality conditions for *ee* and *ue* workers give $u'(c_t^{ee}) = u'(c_t^{ue}) = 1$, so that $c_t^{ee} = c_t^{ue} = u'^{-1}(1) = c^e$. Then, their intertemporal optimality conditions, combined with the budget constraints of *eeu* and *ueu* workers, give

$$1 = \beta\pi R_t + \beta(1-\pi) R_t E_t u'(a_t^{se} R_t - T_{t+1}), \quad s = u, e,$$

so that $a_t^{ee} = a_t^{ue} = a_t^e$ and $c_t^{eeu} = c_t^{ueu} = c_t^{eu}$. Hence the economy with partial risk sharing nests our baseline model as a special case.

Appendix B. Liquidity-constrained Entrepreneurs

B.1. *Proof of Proposition 2*

We must first derive the dynamic system characterising the entrepreneurial equilibrium under the joint conjecture that entrepreneurs are always borrowing-constrained while employed households never are, and then derive from the steady-state relations the range of debt levels compatible with this joint conjecture. Equations (38) and (40) give:

$$\tilde{c}_t^e = u'^{-1}(w_t^{-1}), \quad c_t^f = u'^{-1}[\beta w_t^{-1} E_t(w_{t+1}^{-1})] \tag{B.1}$$

Substituting (B.1) into (43), the goods-market equilibrium can be written as:

$$u'^{-1}(w_t^{-1}) + (1-\theta) u'^{-1}[\beta w_t^{-1} E_t(w_{t+1}^{-1})] + (2-\theta) G_t = (1-\theta) l_{t-1}^f \tag{B.2}$$

Substituting (8), (41) and (B.1) into the budget constraint of *f*-households, (36), gives:

$$u'^{-1}[\beta w_t^{-1} E_t(w_{t+1}^{-1})] + w_t l_t^f = (2-\theta)(B_t - G_t + \Gamma T_t) + \frac{\delta[1 + (1-\theta) R_t]}{R_t}. \tag{B.3}$$

Finally, substituting (B.1) into (39), the Euler equations for employed households is:

$$w_t^{-1} = \beta R_t \{\theta E_t(w_{t+1}^{-1}) + (1-\theta) E_t[\beta w_{t+1}^{-1} E_{t+1}(w_{t+2}^{-1})]\} \tag{B.4}$$

Since shocks are small by assumption, the dynamic system just derived is an equilibrium if, in the steady state, all employed households hold positive assets at the end of the current period [which, from (23), is ensured by $B > 0$] and entrepreneurs are always borrowing-constrained, that is, $u'(c^f) > \beta R u'(\tilde{c}^e)$. From (B.1), this latter condition is equivalent to $wR < 1$. Now, the steady state counterpart of (B.4) gives:

$$w = \beta^2 (1-\theta) R / (1 - \beta\theta R), \tag{B.5}$$

so that $\partial w / \partial R > 0$. Substituting (B.5) into the inequality $wR < 1$, we find that entrepreneurs are borrowing-constrained if and only if $R < 1/\beta$.

We may now compute B^{**}, the unique upper debt level ensuring that $R \in (0, 1/\beta)$ whenever $B \in (0, B^{**})$. First, use the facts that $G = 0$ and $T = B(R-1)$ to write the steady-state counterparts of (B.2) and (B.3) as follows:

$$wl^f = \frac{wu'^{-1}(w_t^{-1})}{1-\theta} + wu'^{-1}(\beta w^{-2}),$$

$$wl^f = \left(B + \frac{\delta}{R}\right)[1 + (1-\theta)R] - u'^{-1}(\beta w^{-2}).$$

Equating the two, using (B.5) and rearranging, we can write steady-state public debt as:

$$\tilde{B}(R) = \left(\frac{R}{1/R + 1 - \theta}\right)\left[\frac{\beta^2(1-\theta)}{1-\beta\theta R}\right]^2\left[\frac{u'^{-1}(w^{-1})}{(1-\theta)w} + \frac{u'^{-1}(\beta w^{-2})}{w} + \frac{u'^{-1}(\beta w^{-2})}{w^2}\right] - \frac{\delta}{R}, \tag{B.6}$$

where w is itself a function of R [see (B.5)]. The term $-\delta/R$ in (B.6) is continuously increasing in R over $(0, \infty)$. The terms inside the first two pairs of large brackets in (B.6), as well as w in (B.5), are all continuously increasing in R over $[0, 1/\theta\beta)$. Hence, if the term inside the third pair of large brackets is non-decreasing in w, then $\tilde{B}(R)$ will be continuous and increasing in R over $(0, 1/\theta\beta)$; we now show that this is the case provided that $\sigma(c) \leq 1$ for all c. By making use of (B.1), we can compute the following derivatives:

$$\frac{\partial}{\partial w}\left[\frac{u'^{-1}(w^{-1})}{w}\right] = \frac{\tilde{c}^e}{w^2}\left[\frac{1}{\sigma(\tilde{c}^e)} - 1\right], \quad \frac{\partial}{\partial w}\left[\frac{u'^{-1}(\beta w^{-2})}{w}\right] = \frac{c^f}{w^2}\left[\frac{2}{\sigma(c^f)} - 1\right],$$

$$\text{and } \frac{\partial}{\partial w}\left[\frac{u'^{-1}(\beta w^{-2})}{w^2}\right] = \frac{2c^f}{w^3}\left[\frac{1}{\sigma(c^f)} - 1\right].$$

All three are non-negative if $\sigma(c) \leq 1$, implying that $\tilde{B}(R)$ is continuously increasing over $(0, 1/\theta\beta)$. With $\delta \geq 0$, the lower bound for R consistent with $\tilde{B} > 0$ is $\tilde{R}_{\min}$ that solves $\tilde{B}(R) = 0$, and by construction we have that $\lim_{R \to \tilde{R}_{\min}} \tilde{B}(R) = 0$ and $\tilde{R}_{\min} < 1/\beta$. Moreover, equation (B.6) implies that $\lim_{R \to 1/\theta\beta} \tilde{B}(R) = \infty$. Thus, we may compute the joint condition on (B, δ) stated in Proposition 2 by evaluating $\tilde{B}(R)$ at $R = 1/\beta$.

As in the model with liquidity-constrained households, one may also compute the equivalent maximum steady-state public debt–output ratio consistent with the bindingness of the borrowing constraint by evaluating steady-state output Y at $R = 1/\beta$. At $R = 1/\beta$, we have $w = \beta$ [see (B.5)] and hence $\tilde{c}^e = c^f = u'^{-1}(\beta^{-1})$ [see (B.1)]. In this situation, the market clearing condition (43) gives $Y = u'^{-1}(\beta^{-1})$ and an upper value for the ratio of:

$$\left(\frac{B}{Y}\right)_{R=1/\beta} = \frac{\beta^2}{\beta + 1 - \theta} + \frac{\beta}{1-\theta}.$$

B.2. *Dynamics and Stability*

The dynamic system characterising the behaviour of the entrepreneurial model is derived as follows. First, substitute the linear counterparts of (9) and (B.1) and into the linearised versions of (8) and (B.2)–(B.4). The latter equations then form a four-dimensional expectational dynamic system with forcing terms G_t and T_t^c and vector of unknowns:

$$X_t = \begin{bmatrix} B_t & l_t^f & R_t & w_t \end{bmatrix}'$$

This system can be solved numerically for its auto-regressive representation using standard methods once values have been assigned to all deep parameters of the model and to the target

debt level B. [Here again the latter is chosen so as to generates a steady state value of R of 1.01, but equation (B.6) rather than (A.2), is used.] Finally, total private consumption is $C_t = (1-\Gamma)\tilde{c}_t^e + \Gamma c_t^f$, with $\tilde{c}_t^e$ and c_t^f given by (B.1), and aggregate output is $Y_t = \Gamma l_{t-1}^f$. For our baseline parameters, the stationarity requirement is $\phi > \phi_{\min} \simeq 0.134$.

B.3. *Long Projects*

Let us here assume that $\delta = 0$ and that all parameters apart from the length of projects are at their baseline value. For any value of τ, there are *ee* entrepreneurs and *fe* entrepreneurs in the economy, with budget constraints (34)–(35) and optimal labour supply and asset demand choices characterised by (38)–(39). However, values of τ higher than one imply that there are several types of entrepreneurs running a project, as many as the number of periods that a project lasts. For the sake of conciseness we only show how to construct the equilibrium when $\tau = 2$ here, but the approach can be applied straightforwardly to higher values of τ.

With two-period projects there are two types of active entrepreneurs, those who are currently starting a project and those who did so in the previous period. Let us call the former 'f entrepreneurs' (by analogy with the $\tau = 1$ case) and the latter 'ff entrepreneurs'. Under C1–C2, their budget constraints are as follows:

$$f : c_t^{ef} + w_t l_t^f = \tilde{a}_{t-1}^e R_{t-1} - T_t, \quad ff : c^{ff} + w_t l_t^{ff} = l_{t-1}^f - T_t.$$

In short, these two equations indicate that entrepreneurs who meet a project opportunity entirely liquidate their asset wealth to invest in it, and will be using the implied output to re-invest in the project (and consume) in the next period; we may then check numerically that when B is sufficiently small, then $w_t R_t < 1$, so that these entrepreneurs would like to borrow rather hold positive assets at the end of every period (that is, the economy is liquidity-constrained). There are now two optimal labour demand conditions [rather than the unique condition (40) in the one-stage case], depending on the stage of the project:

$$w_t u'(c_t^f) = \beta E_t u'(c_{t+1}^{ff}), \quad w_t u'(c_t^{ff}) = \beta E_t u'(\tilde{c}_{t+1}^e).$$

Finally, given that entrepreneurs not running a project will meet a project opportunity with probability θ in the next period and, in this case, run their project for exactly two periods, the (asymptotic) shares of each type of entrepreneur in the economy are:

$$\tilde{\omega}^{ff} = \tilde{\omega}^f = \tilde{\omega}^{fe} = (1-\theta)/(3-2\theta), \quad \tilde{\omega}^{ee} = \theta/(3-2\theta),$$

while the total number of entrepreneurs running a project is $\tilde{\omega}^{ef} + \tilde{\omega}^{ff} = 2(1-\theta)/(3-2\theta)$ here. Total output at date t results from the labour inputs of the two types of active entrepreneurs in the previous period, $\tilde{\omega}^f l_{t-1}^f + \tilde{\omega}^{ff} l_{t-1}^{ff}$. Hence, the model is closed once the following market-clearing conditions are imposed in bonds and goods markets:

$$(1 - \tilde{\omega}^f - \tilde{\omega}^{ff}) a_t = B_t,$$
$$(1 - \tilde{\omega}^f - \tilde{\omega}^{ff}) \tilde{c}_t^e + \tilde{\omega}^f c_t^f + \tilde{\omega}^{ff} c_t^{ff} + G_t = \tilde{\omega}^f l_{t-1}^f + \tilde{\omega}^{ff} l_{t-1}^{ff}.$$

USING E-ANNOTATION TOOLS FOR ELECTRONIC PROOF CORRECTION

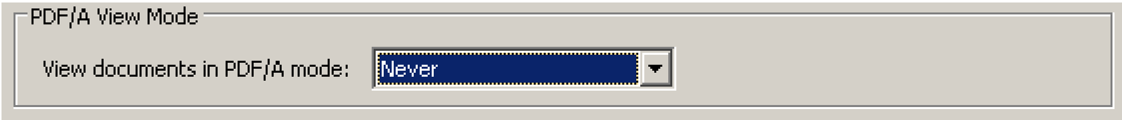
Required Software

Adobe Acrobat Professional or Acrobat Reader (version 7.0 or above) is required to e-annotate PDFs. Acrobat 8 Reader is a free download: http://www.adobe.com/products/acrobat/readstep2.html

Once you have Acrobat Reader 8 on your PC and open the proof, you will see the Commenting Toolbar (if it does not appear automatically go to Tools>Commenting>Commenting Toolbar). The Commenting Toolbar looks like this:

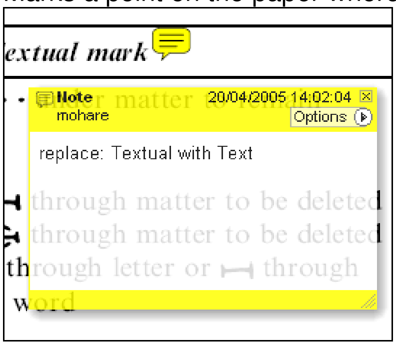
If you experience problems annotating files in Adobe Acrobat Reader 9 then you may need to change a preference setting in order to edit.

In the “Documents” category under “Edit – Preferences”, please select the category ‘Documents’ and change the setting “PDF/A mode:” to “Never”.



Note Tool — For making notes at specific points in the text

Marks a point on the paper where a note or question needs to be addressed.

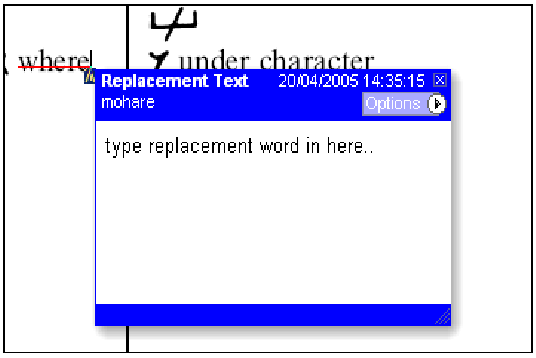


How to use it:

1. Right click into area of either inserted text or relevance to note
2. Select Add Note and a yellow speech bubble symbol and text box will appear
3. Type comment into the text box
4. Click the X in the top right hand corner of the note box to close.

Replacement text tool — For deleting one word/section of text and replacing it

Strikes red line through text and opens up a replacement text box.

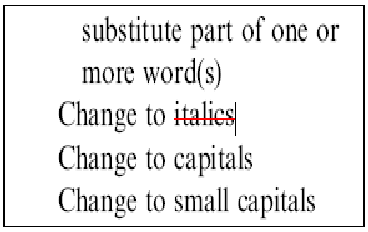


How to use it:

1. Select cursor from toolbar
2. Highlight word or sentence
3. Right click
4. Select Replace Text (Comment) option
5. Type replacement text in blue box
6. Click outside of the blue box to close

Cross out text tool — For deleting text when there is nothing to replace selection

Strikes through text in a red line.



How to use it:

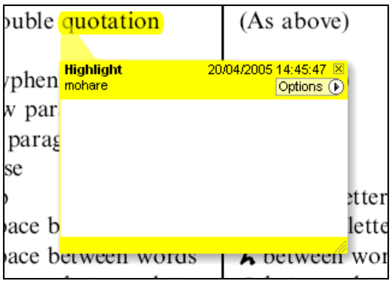
1. Select cursor from toolbar
2. Highlight word or sentence
3. Right click
4. Select Cross Out Text

Approved tool — For approving a proof and that no corrections at all are required.

How to use it:
1. Click on the Stamp Tool in the toolbar
2. Select the Approved rubber stamp from the 'standard business' selection
3. Click on the text where you want to rubber stamp to appear (usually first page)

Highlight tool — For highlighting selection that should be changed to bold or italic.

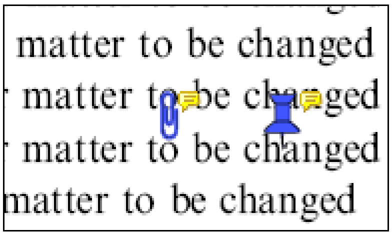
Highlights text in yellow and opens up a text box.



How to use it:
1. Select Highlighter Tool from the commenting toolbar
2. Highlight the desired text
3. Add a note detailing the required change

Attach File Tool — For inserting large amounts of text or replacement figures as a files.

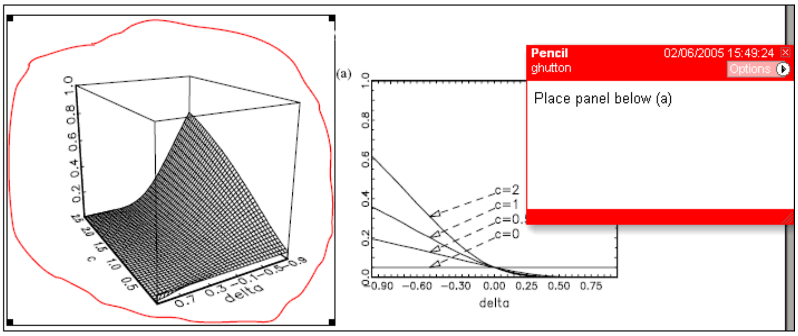
Inserts symbol and speech bubble where a file has been inserted.



How to use it:
1. Click on paperclip icon in the commenting toolbar
2. Click where you want to insert the attachment
3. Select the saved file from your PC/network
4. Select appearance of icon (paperclip, graph, attachment or tag) and close

Pencil tool — For circling parts of figures or making freeform marks

Creates freeform shapes with a pencil tool. Particularly with graphics within the proof it may be useful to use the Drawing Markups toolbar. These tools allow you to draw circles, lines and comment on these marks.



How to use it:
1. Select Tools > Drawing Markups > Pencil Tool
2. Draw with the cursor
3. Multiple pieces of pencil annotation can be grouped together
4. Once finished, move the cursor over the shape until an arrowhead appears and right click
5. Select Open Pop-Up Note and type in a details of required change
6. Click the X in the top right hand corner of the note box to close.

Help

For further information on how to annotate proofs click on the Help button to activate a list of instructions: