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Technical Appendix to

RATIONAL AND NEAR-RATIONAL BUBBLES WITHOUT DRIFT

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Appendix: Derivations and Proofs

A.1. *Separating Consumption from Dividends*

The Lucas (1978) model implies $c_t = d_t$ for all t. This Section outlines a version of the model that allows $c_t \neq d_t$. The agent's first-order condition is

$$\frac{p_t}{d_t} = \mathrm{E}_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\alpha}\left(\frac{d_{t+1}}{d_t}\right)\left(\frac{p_{t+1}}{d_{t+1}} + 1\right)\right]. \tag{A.1}$$

The separate growth rates of consumption and dividends are now given by

$$\log(c_t/c_{t-1}) \equiv x_t^c = \bar{x}^c + \rho\left(x_{t-1}^c - \bar{x}^c\right) + \varepsilon_t, \ \ \varepsilon_t \sim \mathrm{N}\left(0, \sigma_\varepsilon^2\right), \quad |\rho| < 1, \tag{A.2}$$

$$\log(d_t/d_{t-1}) \equiv x_t^d = \bar{x}^d + \gamma\left(x_t^c - \bar{x}^c\right) + v_t \quad v_t \sim \mathrm{N}\left(0, \sigma_v^2\right), \tag{A.3}$$

where v_t is uncorrelated with ε_t. As before, consumption growth is described by a univariate AR(1) process. Deviations of dividend growth from consumption growth are governed by the parameters $\bar{x}^d$, γ and σ_v^2. The original Lucas model with $c_t = d_t$ can be recovered by setting $\bar{x}^d = \bar{x}^c$, $\gamma = 1$, and $\sigma_v = 0$. For the model with $c_t \neq d_t$, these parameters are calibrated to match three moments:

(*i*) the unconditional mean of dividend growth $\mathrm{E}[\log\ (d_t/d_{t-1})]$,
(*ii*) the contemporaneous correlation between dividend growth and consumption growth $\mathrm{corr}(x_t^d, x_t^c)$, and
(*iii*) the unconditional variance of dividend growth $\mathrm{var}(x_t^d)$.

The resulting calibration formulas are

$$\bar{x}^d = \mathrm{E}[\log(d_t/d_{t-1})], \tag{A.4}$$

$$\gamma = \mathrm{corr}(x_t^d, x_t^c)[\mathrm{var}(x_t^d)/\mathrm{var}(x_t^c)]^{0.5}, \tag{A.5}$$

$$\sigma_v = [\mathrm{var}(x_t^d) - \gamma^2 \mathrm{var}\ (x_t^c)]^{0.5}. \tag{A.6}$$

The agent's first-order condition can be written in terms of the price–dividend ratio y_t as follows:

$$\begin{aligned} y_t &= \mathrm{E}_t\{\beta\exp[-\alpha x^c_{t+1} + \bar{x}^d + \gamma(x^c_{t+1} - \bar{x}^c) + v_{t+1}](y_{t+1}+1)\}, \\ &= \mathrm{E}_t[\tilde{\beta}\exp(\tilde{\theta}\tilde{x}_{t+1})(y_{t+1}+1)], \end{aligned} \tag{A.7}$$

where

$$\tilde{\beta} \equiv \beta\exp\left(\bar{x}^d - \gamma\bar{x}^c\right), \quad \tilde{\theta} \equiv \gamma - \alpha, \quad \tilde{x}_t \equiv x^c_t + v_t/\tilde{\theta}.$$

Making use of the above definitions, (A.2) and (A.3) yield the following transformed version of (3):

$$\tilde{x}_t = \bar{x}^c + \rho(\tilde{x}_{t-1} - \bar{x}^c) + \omega_t, \; \omega_t \equiv \varepsilon_t + (v_t - \rho v_{t-1})/\tilde{\theta}, \tag{A.8}$$

where $\omega_t \sim \mathrm{N}\left(0, \sigma^2_\omega\right)$, $\sigma^2_\omega = \sigma^2_\varepsilon + (1+\rho^2)\sigma^2_v/\tilde{\theta}^2$, and corr $(\omega_t, \; \omega_{t-1}) = -\rho/(\tilde{\theta}^2\sigma^2_\varepsilon/\sigma^2_v + 1 + \rho^2)$.

Finally, we define $\tilde{z}_t \equiv \tilde{\beta}\exp(\tilde{\theta}\tilde{x}_t)(y_t+1)$ to obtain the following transformed version of (8):

$$\tilde{z}_t = \tilde{\beta}\exp(\tilde{\theta}\tilde{x}_t)(\mathrm{E}_t\tilde{z}_{t+1} + 1). \tag{A.9}$$

Thus, by an appropriate change of variables, (A.8) and (A.9) retain the same basic forms as (3) and (8), with the exception that the innovation ω_t is not iid but instead exhibits serial correlation. However, for small values of ρ, I can make the simplifying assumption that $\mathrm{corr}(\omega_t, \omega_{t-1}) \simeq 0$. With this assumption, all of the article's theoretical results will go through when expressed in terms of the transformed variables.

A.2. *Proof of Proposition 1: Approximate Fundamental Solution*

Iterating ahead the conjectured law of motion for z^{f}_t and taking the conditional expectation yields

$$\mathrm{E}_t z^{\mathrm{f}}_{t+1} = \exp[a_0 + \rho a_1(x_t - \bar{x}) + \tfrac{1}{2}(a_1)^2\sigma^2_\varepsilon]. \tag{A.10}$$

Substituting the above expression into the first-order condition (8) and then taking logarithms yields

$$\begin{aligned} \log(z^{\mathrm{f}}_t) = F(x_t) &= \log(\beta) + \theta x_t \\ &\quad + \log\{\exp[a_0 + \rho a_1(x_t - \bar{x}) + \tfrac{1}{2}(a_1)^2\sigma^2_\varepsilon] + 1\}, \\ &\simeq a_0 + a_1(x_t - \bar{x}), \end{aligned} \tag{A.11}$$

where $a_0 \equiv \mathrm{E}\left[\log\left(z^{\mathrm{f}}_t\right)\right]$ and a_1 are Taylor-series coefficients which are are given by

$$F(\bar{x}) = a_0 = \log(\beta) + \theta\bar{x} + \log\{\exp[a_0 + \tfrac{1}{2}(a_1)^2\sigma^2_\varepsilon] + 1\} \tag{A.12}$$

$$F'(\bar{x}) = a_1 = \theta + \frac{\rho a_1\exp[a_0 + \frac{1}{2}(a_1)^2\sigma^2_\varepsilon]}{\exp[a_0 + \frac{1}{2}(a_1)^2\sigma^2_\varepsilon] + 1}. \tag{A.13}$$

Solving (A.12) for a_0 yields

$$a_0 = \log\left\{\frac{\beta\exp(\theta\bar{x})}{1 - \beta\exp[\theta\bar{x} + \frac{1}{2}(a_1)^2\sigma^2_\varepsilon]}\right\}, \tag{A.14}$$

which can be substituted into (A.13) to yield the following non-linear equation that determines a_1:

$$a_1 = \theta + \rho a_1\beta\exp[\theta\bar{x} + \tfrac{1}{2}(a_1)^2\sigma^2_\varepsilon]. \tag{A.15}$$

Solving (A.15) for a_1 yields the non-linear equation shown in Proposition 1. There are two solutions but only one solution satisfies the condition $\beta\exp[\theta\bar{x}+\frac{1}{2}(a_1)^2\sigma_\varepsilon^2]<1$ such that $\exp(a_0) = \exp[\mathrm{E}\log(z_t^{\mathrm{f}})]>0$.

A.3. *Asset Pricing Moments: Fundamental Solution*

This Section briefly outlines the derivation of (16)–(19).

Equation (16) follows directly from (15) by taking the unconditional expectation of $\log(y_t^{\mathrm{f}})$. We have

$$\log(y_t^{\mathrm{f}})-\mathrm{E}\left[\log(y_t^{\mathrm{f}})\right] = a_1\rho_1(x_t-\bar{x}), \tag{A.16}$$

which implies $\mathrm{var}[\log(y_t^{\mathrm{f}})] = (a_1)^2\rho^2\,\mathrm{var}(x_t)$, as given by (17).

The fundamental equity return can be written as

$$\begin{aligned} R_{t+1}^{\mathrm{f}} &= \left(\frac{y_{t+1}^{\mathrm{f}}+1}{y_t^{\mathrm{f}}}\right)\exp(x_{t+1}),\\ &= \left(\frac{z_{t+1}^{\mathrm{f}}}{\beta\mathrm{E}_t z_{t+1}^{\mathrm{f}}}\right)\exp(\alpha x_{t+1}), \end{aligned} \tag{A.17]}$$

where I have eliminated y_t^{f} using the equilibrium relationship $y_t^{\mathrm{f}} = \mathrm{E}_t z_{t+1}^{\mathrm{f}}$ and eliminated y_{t+1}^{f} using the definitional relationship $y_{t+1}^{\mathrm{f}}+1 = \beta^{-1}\exp(-\theta x_{t+1})z_{t+1}^{\mathrm{f}}$. Substituting in $z_{t+1}^{\mathrm{f}} = \exp[a_0+a_1(x_t-\bar{x})]$ from Proposition 1 and $\mathrm{E}_t z_{t+1}^{\mathrm{f}}$ from (15) and then taking the unconditional expectation of $\log(R_{t+1}^{\mathrm{f}})$ yields (18). I have

$$\log(R_{t+1}^{\mathrm{f}})-\mathrm{E}[\log(R_{t+1}^{\mathrm{f}})] = \alpha(x_{t+1}-\bar{x})+a_1\varepsilon_{t+1}, \tag{A.18}$$

which in turns implies

$$\mathrm{var}[\log(R_{t+1}^{\mathrm{f}})] = \alpha^2\mathrm{var}(x_t)+(a_1)^2\sigma_\varepsilon^2+2\alpha a_1\mathrm{cov}(x_t,\varepsilon_t). \tag{A.19}$$

Substituting for $\mathrm{var}(x_t)$ and $\mathrm{cov}(x_t, \varepsilon_t)$ in the above expression yields (19).

A.4 *Proof of Proposition 2: Continuum of Intrinsic Rational Bubbles*

First consider the case where the agent can make use of the contemporaneous realisation z_t^{b} when forming the rational expectation $\mathrm{E}_t z_{t+1}^{\mathrm{b}}$. Iterating ahead the conjectured law of motion for z_t^{b} by one period and then taking the conditional expectation yields

$$\mathrm{E}_t z_{t+1}^{\mathrm{b}} = z_t^{\mathrm{b}}\exp[\lambda_0+(\rho\lambda_1+\lambda_2)(x_t-\bar{x})+\tfrac{1}{2}(\lambda_1)^2\sigma_\varepsilon^2]. \tag{A.20}$$

Substituting the above expression into the no-arbitrage condition (21) and then taking logarithms yields

$$0 = \log(\beta)+\theta x_t+\lambda_0+(\rho\lambda_1+\lambda_2)(x_t-\bar{x})+\tfrac{1}{2}(\lambda_1)^2\sigma_\varepsilon^2, \tag{A.21}$$

where $\log(z_t^{\mathrm{b}})$ has been cancelled from both sides. For (A.21) to hold, the constant terms and the coefficients on x_t must separately sum to zero. Equilibrium therefore requires

$$\frac{1}{2}(\lambda_1)^2\sigma_\varepsilon^2-\underbrace{(\rho\lambda_1+\lambda_2)}_{-\theta}\bar{x}+\log(\beta)+\lambda_0 = 0, \tag{A.22}$$

$$\theta+\rho\lambda_1+\lambda_2 = 0, \tag{A.23}$$

which represent a system of two equations in three unknown constants λ_0, λ_1, and λ_2. The solutions to (A.22) and (A.23) define a continuum of intrinsic rational bubble equilibria.

Now consider the case where the agent can only make use of the lagged realisation z_{t-1}^{b} when forming $\mathrm{E}_t z_{t+1}^{\mathrm{b}}$. Iterating ahead the conjectured law of motion for z_t^{b} by one period and then substituting out z_t^{b} using the same law of motion yields

$$z_{t+1}^{\mathrm{b}} = z_{t-1}^{\mathrm{b}} \exp\{2\lambda_0 + [\lambda_1(1+\rho) + \lambda_2](x_t - \bar{x}) + \lambda_2(x_{t-1} - \bar{x}) + \lambda_1 \varepsilon_{t+1}\}, \tag{A.24}$$

where I have eliminated $(x_{t+1} - \bar{x})$ using the law of motion for consumption/dividend growth (33). Taking the conditional expectation of the above expression yields

$$\mathrm{E}_t z_{t+1}^{\mathrm{b}} = z_{t-1}^{\mathrm{b}} \exp\left\{2\lambda_0 + [\lambda_1(1+\rho) + \lambda_2](x_t - \bar{x}) + \lambda_2(x_{t-1} - \bar{x}) + \tfrac{1}{2}(\lambda_1)^2 \sigma_\varepsilon^2\right\}. \tag{A.25}$$

Substituting the above expression into the no-arbitrage condition (21) and then taking logarithms yields

$$\begin{aligned} \log\left(z_t^{\mathrm{b}}\right) = {} & \log(z_{t-1}^{\mathrm{b}}) + \log(\beta) + \theta x_t + 2\lambda_0 + [\lambda_1(1+\rho) + \lambda_2](x_t - \bar{x}) \\ & + \lambda_2(x_{t-1} - \bar{x}) + \tfrac{1}{2}(\lambda_1)^2 \sigma_\varepsilon^2, \end{aligned} \tag{A.26}$$

which can be compared to the following expression for the logarithm of the conjectured law of motion:

$$\log\left(z_t^{\mathrm{b}}\right) = \log\left(z_{t-1}^{\mathrm{b}}\right) + \lambda_0 + \lambda_1(x_t - \bar{x}) + \lambda_2(x_{t-1} - \bar{x}). \tag{A.27}$$

Equation (A.26) will coincide exactly with (A.27) when the following equilibriums conditions are satisfied

$$\log(\beta) + 2\lambda_0 - [\lambda_1(1+\rho) + 2\lambda_2]\bar{x} + \tfrac{1}{2}(\lambda_1)^2 \sigma_\varepsilon^2 = \lambda_0 - (\lambda_1 + \lambda_2)\bar{x}, \tag{A.28}$$

$$\theta + \lambda_1(1+\rho) + \lambda_2 = \lambda_1, \tag{A.29}$$

which are isomorphic to the equilibrium conditions (A.22) and (A.23).

A.5. *Asset Pricing Moments: Near-rational Solution*

Starting from the approximate ALM (36), the law of motion of $\Delta\log(z_t)$ can be written as:

$$\Delta\hat{z}_t = (k-1)[\hat{z}_{t-1} - \mathrm{E}(\hat{z}_t)] + m(x_t - \bar{x}), \tag{A.30}$$

where $\hat{z}_t \equiv \log(z_t)$ and $\Delta\hat{z}_t \equiv \log(z_t/z_{t-1})$. The above equation implies:

$$\mathrm{cov}(\Delta\hat{z}_t, x_t) = (k-1)\mathrm{cov}(\hat{z}_{t-1}, x_t) + m\mathrm{var}(x_t). \tag{A.31}$$

From (36), I have $\mathrm{cov}(\hat{z}_{t-1}, x_t) = [\rho m/(1-\rho k)]\,\mathrm{var}(x_t)$, which can be substituted into (A.31) to yield (40) in the text.

The nonlinear ALM for the price–dividend ratio, (33), can be rewritten as follows:

$$\begin{aligned} y_t &= (y_{t-1}+1)\beta \exp\left[b(1+\rho)(x_t - \bar{x}) + \theta x_{t-1} + \frac{1}{2} b^2 \sigma_\varepsilon^2\right], \\ &= (y_{t-1}+1) k \exp\left[\left(\frac{m-\theta}{k}\right)(x_t - \bar{x}) + \theta(x_{t-1} - \bar{x})\right], \end{aligned} \tag{A.32}$$

where I have eliminated b and b^2 using the expressions for the Taylor series coefficients k and m, as given by (37) and (38). Taking logarithms of the above expression yields

$$\begin{aligned}\hat{y}_t &= \log[\exp(\hat{y}_{t-1})+1]+\log(k)+\left(\frac{m-\theta}{k}\right)(x_t-\bar{x})+\theta(x_{t-1}-\bar{x}),\\ &\simeq n_0+n_1[\hat{y}_{t-1}-\mathrm{E}(\hat{y}_t)]+\left(\frac{m-\theta}{k}\right)(x_t-\bar{x})+\theta(x_{t-1}-\bar{x}),\end{aligned} \tag{A.33}$$

where $\hat{y}_t \equiv \log(y_t)$and n_0 and n_1 are Taylor series coefficients. Straightforward computations yield $n_0 = \log\ [k/(1-k)]$ and $n_1 = k$. The unconditional expectation of the above expression yields $\mathrm{E}(\hat{y}_t) = n_0$, as given by (41).

Using (A.33), the unconditional variance can be computed as follows:

$$\begin{aligned}\mathrm{var}(\hat{y}_t) &= \mathrm{E}\Big\{[\hat{y}_t-\mathrm{E}(\hat{y}_t)]^2\Big\},\\ &= \left(\frac{1}{1-k^2}\right)\left[\left(\frac{m-\theta}{k}\right)^2+\theta^2+2\left(\frac{m-\theta}{k}\right)\theta\rho\right]\mathrm{var}(x_t),\\ &\quad+\left[\frac{2(m-\theta)\rho+2\theta k}{1-k^2}\right]\mathrm{cov}(\hat{y}_t,x_t),\end{aligned} \tag{A.34}$$

where $\mathrm{cov}(\hat{y}_t, x_t)$ can also be computed from (A.33).

The equity return is given by

$$\begin{aligned}R_{t+1} &= \left(\frac{z_{t+1}}{\beta\hat{\mathrm{E}}_t z_{t+1}}\right)\exp(\alpha x_{t+1})\\ &= \frac{z_t^k\bar{z}^{1-k}\exp[m(x_{t+1}-\bar{x})+\alpha x_{t+1}]}{\beta z_{t-1}\exp\left[b(1+\rho)(x_t-\bar{x})+\frac{1}{2}b^2\sigma_\varepsilon^2\right]},\end{aligned} \tag{A.35}$$

where I have substituted in the approximate ALM (36) and the subjective expectation (30). Taking the unconditional expectation of $\hat{R}_{t+1} \equiv \log(R_{t+1})$ yields (42). From (A.35), I have

$$\begin{aligned}\hat{R}_{t+1}-\mathrm{E}\big(\hat{R}_{t+1}\big) &= \underbrace{k[\hat{z}_t-\mathrm{E}(\hat{z}_t)]}_{k^2[\hat{z}_{t-1}-E(\hat{z}_t)]+km(x_t-\bar{x})}-[\hat{z}_{t-1}-\mathrm{E}(\hat{z}_t)]\\ &\quad+(m+\alpha)(x_{t+1}-\bar{x})-b(1+\rho)(x_t-\bar{x}),\end{aligned} \tag{A.36}$$

which can be used to compute an analytical expression for $\mathrm{var}\big(\hat{R}_{t+1}\big)$.

From (43), the law of motion for the percentage forecast error is given by

$$\begin{aligned}err_{t+1}-\mathrm{E}(err_{t+1}) &= -\big(1-k^2\big)[\hat{z}_{t-1}-\mathrm{E}(\hat{z}_t)]\\ &\quad+[km+m\rho-b(1+\rho)](x_t-\bar{x})+m\varepsilon_{t+1},\end{aligned} \tag{A.37}$$

where I have eliminated $[\hat{z}_t - \mathrm{E}(\hat{z}_t)]$ using the approximate ALM (36). Equation (A.37) is used to compute $\mathrm{Corr}(err_{t+1}, err_t) = \mathrm{Cov}(err_{t+1}, err_t)/\mathrm{var}(err_{t+1})$ and $RMSPE = [\mathrm{E}(err_{t+1}^2)]^{0.5}$ which are plotted in Figure 3.