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Technical Appendix to

THE REGULATION OF ENTRY AND AGGREGATE PRODUCTIVITY

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Appendix A. Formal Statements of Results and Proofs

For any point of the state space, a firm's labour demand, output and profits, and aggregate labour demand and output can be obtained by static optimisation. Firm value then is given by the functional equation

$$W(v,s,\omega,\xi) = \sup_{x\in\{0,1\}} \{\pi(s,\omega,\xi) + \beta[x+(1-x)\delta]W_x + \beta(1-x)(1-\delta)\mathrm{E}[W(v,s',\omega,\xi)|s]\}, \tag{9}$$

where x is the value taken on by the exit policy function $X(v,\ s,\ \omega,\ \xi)$ ($x = 1$ means exit), and $\pi(\cdot)$ is the profit function resulting from static optimisation.

Lemma 1. *There is a unique firm value function $W(\cdot)$ that satisfies* (9). *The exit policy function $X(\cdot)$ is single-valued and lets firms attain the supremum in* (9).

Proof. Proof is by applying Theorem 9.12 from Stokey and Lucas (1989). Assumption 9.1 trivially holds. Since the expectation of s is finite and v^* is finite by Assumption $3'$, total returns are bounded, and Assumption 9.2 holds. Conditions (a) and (b) of Theorem 9.12 are also fulfilled if v^* is finite. □

Corollary 2. *The firm value function $W(\cdot)$ is continuous, strictly increasing in v and in s, and strictly decreasing in ω. For given v, it is bounded.*

This follows from the properties of the profit function; by Theorems 9.7 and 9.11 in Stokey and Lucas (1989) they carry over to the value function. Boundedness then follows from the fact that $\mathrm{E}(s'|s)$ is well-defined and finite for all s.

Corollary 3. *For $c^f > 0$ and under Assumption 1, there is a unique exit trigger $s_x(v,\ \omega,\ \xi) \equiv \{s \text{ s.t. } \mathrm{E}[W(v,\ s',\ \omega,\ \xi)|s] = W_x\}$. Hence, the exit policy function X is single-valued; it takes value 1 (exit) for $s < s_x$ and value 0 for $s \geq s_x$. The exit trigger $s_x(\cdot)$ is strictly decreasing in v, strictly increasing in ω, and continuous in both.*

Proof. Firms exit whenever the expected value of continuing is smaller than the value of exiting:

$$\mathrm{E}[W(v,s',\omega,\xi)|s] < W_x = 0, \tag{10}$$

where the value of exit W_x is zero due to the zero net value of entry condition (6). Since $\mathrm{E}(s'|s)$ increases in s by Assumption 1 and because firm value increases in s by Corollary 2, the left-hand side (LHS) of (10) is strictly increasing in s. Moreover, given any $c^f > 0$, there is an s so low that

expected value of continuing is negative and an s so high that it is positive. Then there is a unique s_x such that an equality replaces the inequality in (10). Firms exit whenever $s < s_x$. The properties of s_x follow from the properties of the value function. □

To ensure that the condition for optimal technology choice (5) is well-defined, it is necessary to show that the value function is differentiable with respect to v. For this, it first has to be shown that expected firm life is finite. This is also crucial for a stationary equilibrium. Moreover, the result highlights that it is not necessary that the exogenous breakdown rate δ be strictly positive for the results to go through.

Lemma 4. *Given the specification of the stochastic process for s in* (2)*, the lifetime T of a firm is finite for all v with probability* 1. *It has a well-defined pre-entry expectation* $\bar{T}$ *that is the same for all firms.*

Proof. Proof is easiest by reasoning in terms of the properties of Markov processes. Define the set $S_x = \{s \in S : s < s_x\}$. Once a firm draws an $s \in S_x$, it exits, so S_x is an ergodic set. Because the firm's productivity innovation ϵ has positive variance, there are $s \geq s_x$ such that $G_v(s_x|s) > 0$, i.e. with a positive probability of exiting in the next period. Hence, the set $\{s \in S : s \geq s_x\}$ is transient. Then, by Theorem 5.6 in Doob (1953), s can remain outside S_x for a finite time only with probability 1. Moreover, the probability of remaining in the transient set decreases at a geometric rate. As a consequence, expected firm life is finite with probability 1. This implies that it has a well-defined expectation $\bar{T}$. As all firms choose the same v, it is the same for all firms from a pre-entry perspective. □

Lemma 5. *If Assumptions* 3 *and* 3′ *hold, the expected value of entry* W^e *is continuously differentiable in v in a neighbourhood D of* v^*, *with* $W^e_v > 0$ *for all* $v \in D$.

Proof. Gross firm value can be expressed as a sum of current and future profits, weighted by their conditional probabilities. Since firm lifetime is finite with probability 1 and has a well-defined pre-entry expectation (Lemma 4), this sum is finite. All the summands are convex, so their sum is convex. Just as gross firm value, the entry investment function is also continuous, monotonically increasing, and convex in v. So net entry value is also continuous in v. To infer its shape, consider first its limits. Both are negative: the net value of entry goes to minus infinity as v goes to plus infinity by Assumption 3′, and it goes to a negative number as v goes to minus infinity because $\lim_{v \to -\infty} c^e(v) \geq 0 > -c^f = \lim_{v \to -\infty} W(v)$ by Assumption 3 and by optimal exit. Disregard the case where net entry value is always smaller than at the limit where v goes to minus infinity, since then there would be no v with positive entry, hence no entry in equilibrium. Instead focus on the case where there is some v with higher net entry value than in the limits. In that case, there is some v^* that yields the maximum net entry value. Since net entry value is a continuous function of v, it must be concave around v^*. As v is a control variable and constant over a firm's life, its domain can be limited to that concave part. Call its domain D. So the (relevant part of) the net entry value function is concave. Hence, it can be written as a weighted sum of concave net period return functions. Having obtained this, it follows from Theorem 9.10 in Stokey and Lucas (1989) that the expected net value of entry W^e is differentiable with respect to v. The derivative is positive by Corollary 2. Moreover, the firm's technology choice problem is concave, and the first order condition (5) is sufficient. □

The central result for a unique equilibrium then is:

Proposition 6. *Under the assumptions made, equilibrium condition* (*i*) *is fulfilled by a unique finite pair* (v^*, ω^*) *for a given* ξ.

Proof. Equilibrium existence has been shown in the main text. Finiteness of v^* follows from Assumption 3′. Uniqueness of the equilibrium pair (v^*, w^*) follows from the following reasoning. Both expected gross value of entry and the entry investment cost function are convex in v. The

two do not coincide (their limits differ). By (5) and (6), equilibrium occurs at a tangency. Since two convex univariate functions can have at most one tangency, the equilibrium is unique. □

With an expression for the exit trigger, and v^* and ω^* consistent with positive entry in hand, it remains to determine a firm distribution μ and a measure of firms $\bar{\mu}$ consistent with a stationary equilibrium. For obtaining the distribution, there are two crucial ingredients. First, as shown in the main text, all entrants in a given period adopt the same technology. For a stationary equilibrium, clearly, this is constant over time so that we can fix v at v^* and consider $\mu(s)$. Second, there is a one-to-one mapping from the exit trigger s_x to entry mass M. This follows from the fact that given a stationary $\mu(s)$, the total measure of firms has to be constant, and hence the measure of exiting firms $\mu(s < s_x)$ has to equal the measure of entering firms M. Since expected firm life is finite (Lemma 4), this can be achieved. The firm distribution in a stationary equilibrium then is a fixed point of the operator T defined by

$$(T\mu)(s) = \int_{s_x(\mu)}^{\infty} (1-\delta)\mu(u) g_{v^*}(s|u)\mathrm{d}u + Mh_{v^*}(s), \tag{11}$$

i.e. a μ such that $(T\mu)(s') = \mu(s')$. Fixed-point arguments as given in Stokey and Lucas (1989) do not apply easily in this case because, due to entry and exit, the transition function for $\mu(s)$ is not monotone: Every period, low-productivity firms perish and are replaced by more productive ones, with only the remaining firms' productivity following a monotone process. However, the conditions for the existence of a unique stationary equilibrium with positive entry and exit derived in Hopenhayn (1992, equation 12) carry over exactly to the present case. The result that v^* is finite and the fact that the profit function is multiplicatively separable in productivity and the wage are sufficient for this.

For comparative statics, it is necessary to know how W_v^e interacts with ω and ξ. Unfortunately, general statements about second derivatives of value functions are hard to make, but the next two results establish that W_v^e falls in the wage and rises in ξ.

Lemma 7. *W_v^e is strictly decreasing in ω.*

Proof. Write expected gross value of entry as

$$W^e(v, \omega, \xi) = \int_S h_v(s^0) W(v, s^0, \omega, \xi)\mathrm{d}s^0.$$

Its derivative with respect to v is

$$\frac{\partial W^e(v,\omega,\xi)}{\partial v} = \int_S h_v(s^0)\frac{\partial W(v, s^0, \omega, \xi)}{\partial v}\mathrm{d}s^0 + \int_S \frac{\partial h_v(s^0)}{\partial v} W(v, s^0, \omega, \xi)\mathrm{d}s^0.$$

Now consider an $\omega' > \omega$. The second integral becomes smaller because W decreases in ω. The first integral is a weighted average of W_v for $s^0 \geq s_x(v, \omega, \xi)$ (continue), which is positive, and for $s^0 < s_x(v, \omega, \xi)$, which is zero. Increasing the wage raises s_x and thereby puts more weight on the second term, hence the first integral decreases in ω, too. As a result, W_v^e falls in ω. □

Lemma 8. *W_v^e is strictly increasing in ξ.*

Proof. First, show that increases in ξ raise profits for high s and reduce them for low s. The derivative of log profits with respect to ξ is

$$\frac{\partial \ln \pi(s,\omega,\xi)}{\partial \xi} = \ln \exp(s) - \ln\frac{\xi\omega}{\xi-1} = \ln\exp(s) - \ln \bar{s}^{1/(\xi-1)}$$

using $Q = [\xi/(\xi-1)]\, wN = \bar{s}^{1/(\xi-1)}N$, where $\bar{s} = \int \mu(s)\exp(s)^{\xi-1}\mathrm{d}s$. As this has the same sign as $\partial\pi/\partial\xi$, it implies that for firms with productivity above aggregate productivity $\bar{s}^{1/(\xi-1)}$, increases in ξ raise profits, while they lower them for firms with $\exp(s)$ below aggregate productivity. As a

consequence, π is steeper in s for higher ξ, i.e., $\partial\pi/\partial s$ strictly increases in ξ. (The same relationships hold for output and employment.)

As choosing a higher v implies higher expected s, and W^e is differentiable with respect to v, higher ξ then also implies strictly higher W^e_v. $\square$

Appendix B. Homogeneous Firm Model

The production function is

$$y_i = e^{s_i} n,$$

where s_i is constant over time for a given firm. The optimal choice of p then is $[\xi/(\xi - 1)][\omega/\exp(s_i)]$, implying output

$$q(s_i) = \left[\frac{\xi}{\xi - 1}\frac{\omega}{\exp(s_i)}\right]^{-\xi} P^{\xi} Q,$$

labour demand

$$n(s_i) = \left(\frac{\xi\omega}{\xi - 1}\right)^{-\xi} \exp(s_i)^{\xi-1} P^{\xi} Q$$

and profits

$$\pi(s_i) = \frac{1}{\xi - 1}\left(\frac{\xi}{\xi - 1}\right)^{-\xi} \exp(s_i)^{\xi-1}\omega^{1-\xi} P^{\xi} Q - c^f$$

for all firms. With an exogenous exit probability of δ for each firm each period, firm value then is

$$V(s_i) = \frac{\pi(s_i)}{\rho},$$

where $\rho = 1 - \beta(1 - \delta)$. Firms choose s upon entry, at cost $c^e(s) = k_1 e^{k_2 s} + k_3$. As that choice depends on aggregate variables only, and those are constant over time, all entrants at all t choose the same s, so the i subscript can be dropped. Entrants' optimal choice of s involves setting

$$V'(s) = \frac{\pi'(s)}{\rho} = c^{e\prime}(s).$$

Denote this condition by FOC. At the same time, with free entry, net value of entry must be zero in equilibrium, i.e.,

$$V(s) = c^e(s).$$

Denote this condition by NEC. Combining these conditions yields

$$e^s = \left[\frac{c^f/\rho + k_3}{k_1}\frac{\xi - 1}{k_2 - (\xi - 1)}\right]^{\frac{1}{k_2}}$$

as the optimal choice of s.

There are two additional conditions. Normalising the price index

$$P = \left(\int_i p_i^{1-\xi}\right)^{\frac{1}{1-\xi}}$$

to unity implies

$$\omega = \frac{\xi - 1}{\xi} \exp(s) B^{1/(\xi-1)},$$

where B denotes the number of firms. At the same time, from labour market clearing ($N = Bn$),

$$B = \left(\frac{\xi}{\xi - 1}\omega\right)^{\xi} \frac{1}{\exp(s)^{\xi-1} Q}$$

where $N \equiv 1$ is labour supply. Combining this with the expression for ω yields aggregate output

$$Q = \frac{\xi}{\xi - 1}\omega.$$

Substituting this into the profit function eliminates the dependency on Q and allows solving NEC for the wage at the optimal choice of s. This yields

$$\omega = \left[e^{s(\xi-1)} \frac{\tilde{\xi}}{c^f + \rho c^e}\right]^{\frac{1}{\xi-2}},$$

where $\tilde{\xi} = [1/(\xi - 1)][\xi/(\xi - 1)]^{1-\xi}$. From this follow n, aggregate labour demand, and the number of firms B.

To calibrate the model, set ξ, β and δ to 3, 0.96 and 4.5%, respectively, set c^e/c^f as in the heterogeneous firm model and set c^f to match the average establishment size of 15.8. As in the main text, k_1 and k_2 can then be backed out from FOC and NEC. Also set $\xi(B)$ as in the main text. For results, see Table 5.

Appendix C. Model with Bargaining

Differences with respect to the benchmark model are as follows. The labour market is not competitive but the distribution of rents is determined by bargaining. Concretely, assume every period, the firm and a firm-level union set the wage to maximise

$$(1 - \gamma) \ln[\pi(\omega, s) + c^f] + \gamma \ln(\omega - b) + \gamma \ln n(\omega, s),$$

where γ represents workers' bargaining power and b their outside option, and the firm's outside option at zero employment is $-c^f$. The firm then chooses employment as a function of the bargained wage. This corresponds to Nickell and Andrews (1983) 'right-to-manage' model.

The firm's labour demand and profit functions are as in the benchmark model. Solving the bargaining problem yields a wage

$$\omega = \frac{\sigma + \gamma - 1}{\sigma - 1} b.$$

Instead of being competitively determined, it increases in the outside option b and in the union's bargaining power and decreases as the demand elasticity increases.[1] Goods market clearing implies

$$N = \left(\frac{\sigma}{\sigma - 1}\omega\right)^{\sigma-1} \bar{s}^{-1} = \left(\frac{\sigma}{\sigma - 1}\frac{\sigma + \gamma - 1}{\sigma - 1} b\right)^{\sigma-1} \bar{s}^{-1}.$$

Employment is determined by labour demand and implies

[1] The fact that the wage does not depend on the firm's productivity is due to the constant elasticity of profits with respect to the wage. Bruno and Sachs (1985) use this property to explain real wage rigidity in the face of changes in aggregate productivity.

$$1-u=N\int\mu(s)\left(\frac{\sigma\omega}{\sigma-1}\right)^{-\sigma}\exp(s)^{\sigma-1}Q\mathrm{d}s=N\bar{s}\left(\frac{\sigma\omega}{\sigma-1}\right)^{-\sigma}Q=\frac{\sigma-1}{\sigma}\frac{Q}{\omega}.$$

Imposing $b=\phi Q$,

$$1-u=\frac{\sigma-1}{\sigma\phi}\frac{\sigma-1}{\sigma+\gamma-1}.$$

Unemployment rises in ϕ and in γ. This expression allows setting ϕ to fit u, given γ.

Finally, the firm's value function and exit decision, the free entry condition, optimal technology choice and the evolution of the firm productivity distribution are analogous to the main text.