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TECHNICAL APPENDIX TO ''COMMERCIAL BROADCASTING AND LOCAL CONTENT: CULTURAL QUOTAS, ADVERTISING AND PUBLIC STATIONS''

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A Rationale for the Choice of a Single Station by Advertisers

Suppose there are two broad types of product, *l*-type products and *r*-type products, which differ in the following way: the number of units of an *l*-good sold to a consumer at location $s \in [0, 1]$ on hearing $a_L^l(a_R^l)$ adverts for it at $L(R)$ is $\lambda_s a_L^l(\lambda_s a_R^l)$ and the number of units of an *r*-good sold to the consumer on hearing $a_L^r(a_R^r)$ adverts for it at $L(R)$ is $(1-\lambda_s)a_L^r[(1-\lambda_s)a_R^r]$. That is, while demand by all consumers for a good is increasing in the amount of advertising heard, advertising for *l*-goods is more effective, in terms of leading to greater sales, at L than at R because consumers close to L are more susceptible (whereas the reverse is true for *r*-goods.) Note that a given consumer responds equally to a given advert from either station; it is just that consumers at the L end of the spectrum respond more favourably to adverts for *l*-goods than to those for *r*-goods.

Total sales of an *l*-good will then be:

$$\int_0^{s_L} \lambda_s a_L^l \mathrm{d}s + \int_{s_L}^{s_R} (\lambda_s a_L^l + \lambda_s a_R^l)\mathrm{d}s + \int_{s_R}^{1} \lambda_s a_R^l \mathrm{d}s$$

$$= s_L a_L^l + \int_{s_L}^{s_R} \lambda_s (a_L^l + a_R^l)\mathrm{d}s = s_L a_L^l + \frac{1}{2}(R-L)(a_L^l + a_R^l)$$

where $s_L = L + \dfrac{a_R - a_L}{2t(R-L)}$, $s_R = R + \dfrac{a_R - a_L}{2t(R-L)}$ and $a_j = a_j^l + a_j^r$, $j = L, R$. Similarly, R sales will be $(1-s_R)a_R^r + \frac{1}{2}(R-L)(a_L^r + a_R^r)$.

Consider an *l*-type firm. An increase in a_L^l will always yield at least as great an increase in sales as an equal increase in a_R^l. Supposing that there is some fixed cost, G, involved in developing an ad campaign at a station, if the gains from an ad campaign at L exceed G but those from a campaign at R do not (i.e. if $\left[\frac{1}{2}(R-L) + s_L - p_L\right]a_L^l > G > \left[\frac{1}{2}(R-L) - p_R\right]a_R^l$) then an *l*-type firm will advertise only at L (and a similar condition establishes that an *r*-type firm will advertise only at R.)

Under that condition an *l*-type firm seeks to solve the following problem: $\underset{\{a_L\}}{Max}\left[s_L + \frac{1}{2}(R-L) - p_L\right]a_L = \left[L + \dfrac{a_R - a_L}{2t(R-L)} + \frac{1}{2}(R-L) - p_L\right]a_L$ where we have dropped superscripts for clarity. This yields a first-order condition: $a_L = \frac{1}{2}[a_R + t(R-L)(R+L-2p_L)]$. A similar exercise for an *r*-type firm yields the following: $a_R = \frac{1}{2}\{a_L + t(R-L)[2-(R+L)-2p_R]\}$ and solving these jointly yields the first order conditions provided as (13) in the article.

Derivation of Advertising Demands

From (6) in the article we can solve the first-order conditions in (7):

$$a_L x_L = \frac{1}{2t(R-L)}\{[a_R + t(R-L)(R+L)]a_L - a_L^2\}$$
$$a_R x_R = \frac{1}{2t(R-L)}(\{a_L + [2-(R+L)]t(R-L)\}a_R - a_R^2)$$

$$\Rightarrow \frac{\mathrm{d}(a_L x_L)}{\mathrm{d}a_L} = \frac{1}{2t(R-L)}[a_R + t(R-L)(R+L) - 2a_L]$$
$$\frac{\mathrm{d}(a_R x_R)}{\mathrm{d}a_R} = \frac{1}{2t(R-L)}\{a_L + [2-(R+L)]t(R-L) - 2a_R\}$$

$$\Rightarrow a_L = \frac{1}{2}[a_R + t(R-L)(R+L)] - t(R-L)p_L$$
$$a_R = \frac{1}{2}\{a_L + [2-(R+L)]t(R-L)\} - t(R-L)p_R$$

to give, as in the article:

$$\begin{aligned} a_L(L,R,p_L,p_R) &= \frac{1}{3}t(R-L)[2+(R+L)-2(2p_L+p_R)] \\ a_R(L,R,p_L,p_R) &= \frac{1}{3}t(R-L)[4-(R+L)-2(2p_R+p_L)]. \end{aligned} \tag{13}$$

Welfare

Starting with consumer welfare, for a consumer at $s \le s_L$ disutility is $t(L-s)^2 + a_L$ while for $s \ge s_R$ disutility is $t(s-R)^2 + a_R$. For a consumer at $s \in [L, R]$ disutility is $t(\Gamma_s - s)^2 + [\lambda_s a_L + (1-\lambda_s)a_R]$. Hence,

$$\begin{aligned} U &= \int_0^{s_L} [\underline{v} - a_L - t(L-s)^2]\mathrm{d}s + \int_{s_L}^{s_R} \{\underline{v} - [\lambda_s a_L + (1-\lambda_s)a_R] - t(\Gamma_s - s)^2\}\mathrm{d}s \\ &\quad + \int_{s_R}^{1} [\underline{v} - a_R - t(s-R)^2]\mathrm{d}s \\ &= \underline{v} - \int_0^{s_L} [a_L + t(L-s)^2]\mathrm{d}s - \int_{s_L}^{s_R} \{[\lambda_s a_L + (1-\lambda_s)a_R] + t(\Gamma_s - s)^2\}\mathrm{d}s \\ &\quad - \int_{s_R}^{1} [a_R + t(s-R)^2]\mathrm{d}s. \end{aligned} \tag{8}$$

But we can rewrite λ_s when optimally chosen as $(R + B - s)/(R - L)$ where $B \equiv (a_R - a_L)/2t(R-L)$. Hence:

$$\lambda_s a_L + (1-\lambda_s) a_R = \frac{1}{R-L}[Ra_L - La_R + (s-B)(a_R - a_L)]. \tag{A1}$$

Furthermore, $\Gamma_s - s = -B$ so $t(\Gamma_s - s)^2 = tB^2$ so:

$$\begin{aligned} U = \underline{v} - s_L a_L + \left[\frac{t}{3}(L-s)^3\right]\Big|_0^{s_L} - (1-s_R)a_R - \left[\frac{t}{3}(s-R)^3\right]\Big|_{s_R}^{1} \\ - \left\{\frac{s}{R-L}[Ra_L - La_R - B(a_R - a_L)] + tBs^2 - tB^2 s\right\}\Big|_{s_L}^{s_R}. \end{aligned} \tag{A2}$$

Now, $s_R - s_L = R - L$ and $s_R^2 - s_L^2 = R^2 - L^2 + [(a_R - a_L)/t] = (R-L)(R+L) + [(a_R - a_L)/t]$ so

$$\begin{aligned} U &= \underline{v} - s_L a_L - (1-s_R)a_R + \frac{t}{3}[(L-s_L)^3 - L^3] - \frac{t}{3}[(1-R)^3 - (s_R - R)^3] \\ &\quad - \left\{\frac{s_R - s_L}{R-L}[Ra_L - La_R - B(a_R - a_L)] + tB(s_R^2 - s_L^2) - t(R-L)B^2\right\} \\ &= \underline{v} - s_L a_L - (1-s_R)a_R + \frac{t}{3}[(L-s_L)^3 - L^3 + (s_R - R)^3 - (1-R)^3] \\ &\quad - \{[Ra_L - La_R + tB(R-L)(R+L)] - t(R-L)B^2\} \\ &= \underline{v} - s_L a_L - (1-s_R)a_R + \frac{t}{3}[(L-s_L)^3 - L^3 + (s_R - R)^3 - (1-R)^3] \\ &\quad - \left[\frac{1}{2}(a_L + a_R)(R-L) - t(R-L)B^2\right] \\ &= \underline{v} - \left[s_L + \frac{1}{2}(R-L)\right]a_L - \left[1 - s_R + \frac{1}{2}(R-L)\right]a_R \\ &\quad + \frac{t}{3}[(L-s_L)^3 - L^3 + (s_R - R)^3 - (1-R)^3] - [t(R-L)B^2]. \end{aligned} \tag{A3}$$

Let A denote total advertising heard i.e.

$$A \equiv a_L x_L + a_R x_R. \tag{A4}$$

Now, $t(R-L)B^2 = (a_R - a_L)^2/4t(R-L) = 4t(R-L)[1-(R+L)]/81$ from (11). Finally, $s_L = L + B \Rightarrow (L - s_L)^3 = -B^3$ and $s_R = R + B \Rightarrow (s_R - R)^3 = B^3$ so, from our earlier definitions of market shares,

$$U = \underline{v} - A - \frac{t}{3}[L^3 + (1-R)^3] - \frac{4t}{81}(R-L)[1-(R+L)]^2. \tag{A5}$$

Now,

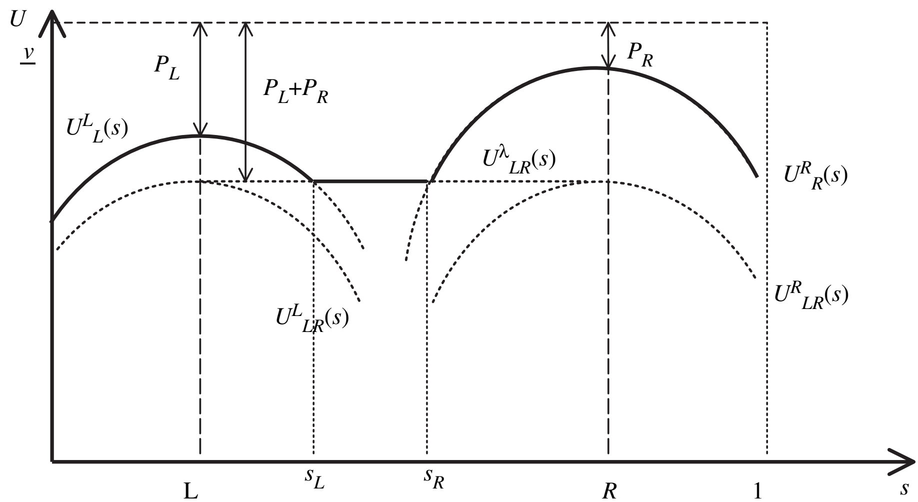
$$\begin{aligned} Y \equiv \pi_L + \pi_R + S_L + S_R &= (p_L a_L - F) + (p_R a_R - F) + (a_L x_L - p_L a_L) + (a_R x_R - p_R a_R) \\ &= A - 2F \end{aligned}$$

Thus, as in the paper,

$$W = \underline{v} - 2F - \frac{t}{3}[L^3 + (1-R)^3] - \frac{4t}{81}(R-L)[1-(R+L)]^2. \tag{A6}$$

The Subscription Case with Fixed Fees: A Benchmark

Suppose a consumer listening to station $j = L, R$ must pay a fixed fee P_j regardless of total usage time. If the consumer chooses to listen to both stations their total expenditure will be $P_L + P_R$ and they will choose the mix between the two to replicate their ideal location. Diagrammatically,



In this Figure the utility from subscribing and listening only to station $j = L, R$ (and paying only P_j) is shown as U_j^j, the utility from subscribing to both (paying $P_L + P_R$) and listening only to j is U_{LR}^j and the utility from subscribing and listening to both with the optimal mix is shown as the horizontal line U_{LR}^{λ}.

For a consumer subscribing and listening to both stations utility is given by: $U = \underline{v} - t[\lambda L + (1-\lambda)R - s]^2 - (P_L + P_R)$ and choosing λ to maximise this yields $\lambda = (R - s)/(R - L)$ which implies that $\lambda L + (1 - \lambda)R = s$. Thus $U_{LR}^{\lambda} = \underline{v} - (P_L + P_R)$. A consumer listening to only one station will subscribe only to that station, hence:

$$U = \underline{v} - \min \begin{cases} t(L - s)^2 + P_L \\ P_L + P_R \\ t(R - s)^2 + P_R. \end{cases}$$

Consequently, a consumer at location s will consume only L if $t(s - L)^2 < P_R$ i.e. if $s < L + (P_R^{1/2}/t) \equiv s_L$ and will consume only R if $t(R - s)^2 < P_L$ i.e. if $s > R - (P_L^{1/2}/t) \equiv s_R$. They will subscribe to and consume both otherwise, i.e. if $s \in [s_L, s_R]$.

Suppose, then, that some consumers subscribe to both stations. Then demand for the L station is $x_L = s_R$ and demand for R is $x_R = 1 - s_L$. Accordingly, $\pi_L = P_L x_L - F = P_L R - (P_L^{3/2})/t - F$ and maximising this over P_L yields:

$$P_L = \left(\frac{2tR}{3}\right)^2.$$

A similar exercise for R yields:

$$P_R = \left[\frac{2t(1-L)}{3}\right]^2$$

and plugging these into our expressions for s_L ands_R gives $s_L^* = 1/3(2-L)$ and $s_R^* = 1/3R$, as noted in the article, which is inconsistent with $L \le R \le 1$ and $s_L < s_R$. The reason is that if some consumers choose to subscribe and listen to both stations then, regardless of locations, the stations cannot be pricing optimally: they should increase prices and squeeze such consumers out.

As a consequence, all consumers subscribe to and listen to only a single station and the model is isomorphic to a standard Hotelling model with non-combinable products. The marginal consumer between L and R is denoted x^* and defined by $t(x-L)^2 + P_L = t(R-x)^2 + P_R$ which solves for $x^* = t(R+L)(R-L) + \Delta/2t(R-L)$ where $\Delta \equiv P_R - P_L$ and demand for each station is $x_L = x^*$ and $x_R = 1 - x^*$. Thus profits are $\pi_j = P_j x_j$ and maximising each profit function over the choice of P_j yields the following price reaction functions:

$$P_L = 1/2[P_R + t(R+L)(R-L)]$$
$$P_R = 1/2\{P_L + t(R-L)[2-(R+L)]\}.$$

From these we can solve for equilibrium prices:

$$P_L = 1/3t(R-L)[2+(R+L)]$$
$$P_R = 1/3t(R-L)[4-(R+L)].$$

These, in turn, yield equilibrium market shares and thence profits:

$$x_L = [2+(R+L)]/6$$
$$x_R = [4-(R+L)]/6$$

$$\pi_L = t(R-L)[2+(R+L)]^2/18 - F$$
$$\pi_R = t(R-L)[4-(R+L)]^2/18 - F$$

as noted in the article. In terms of location choices, then,

$$\frac{\partial \pi_L}{\partial L} = \frac{t}{18}\{2(R-L)[2+(R+L)] - [2+(R+L)]^2\}$$
$$= \frac{t}{18}[2+(R+L)]\{2(R-L) - [2+(R+L)]\} < 0 \,\forall L$$

where the sign follows from $2(R-L) < 2$ and $2 + (R+L) > 2$. Similarly,

$$\frac{\partial \pi_R}{\partial R} = \frac{t}{18}\{[4-(R+L)]^2 - 2(R-L)[4-(R+L)]\}$$
$$= \frac{t}{18}[4-(R+L)][4-(R+L)-2(R-L)] > 0 \,\forall R.$$

Thus the familiar principle of maximal differentiation prevails: $L^* = 0$ and $R^* = 1$. So $x_j^* = 1/2$, $P_j^* = t$ and $\pi_j^* = 1/2t - F$.

Total consumer utility is then:

$$U = \int_0^{x*} [\underline{v} - t(x-L)^2 - P_L]\mathrm{d}x + \int_{x*}^{1} [\underline{v} - t(R-x)^2 - P_R]\mathrm{d}x$$
$$= \underline{v} - x^* P_L - (1-x^*)P_R - \frac{1}{3}t\left[(x-L)^3\Big|_0^{x*} - (R-x)^3\Big|_{x*}^{1}\right]$$

and so welfare is:

$$\begin{aligned} W &= U + \pi_L + \pi_R \\ &= U + (P_L x^* - F) + [P_R(1-x^*) - F] \\ &= \underline{v} - 2F - \frac{1}{3}t[(x^*-L)^3 + L^3 - (R-1)^3 + (R-x^*)^3]. \end{aligned}$$

Evaluating this at $L = 0$ and $R = 1$ (so $x^* = 1/2$) gives $W = \underline{v} - 2F - (t/12)$ in contrast to the level of welfare when locations are chosen to maximise it: at $L = 1/4$ and $R = 3/4$ (so $x^* = 1/2$ again) we obtain the higher level $W = \underline{v} - 2F - (t/48)$.

The Subscription Case With Usage Fees: A Benchmark

Now a consumer located at s listening to both stations in proportion λ at L and $1-\lambda$ at R will derive surplus of: $U = \underline{v} - t[\lambda L + (1-\lambda)R - s]^2 - [\lambda p_L + (1-\lambda)p_R]$. Choosing λ to maximise this yields an optimal mixing ratio of $\lambda_s = \frac{R-s}{R-L} + \frac{p_R - p_L}{2t(R-L)}$ that is, the stations' prices now substitute for advertising levels in our main formulation. From this we can derive the optimal mix for a listener $\Gamma_s = s - \frac{p_R - p_L}{2t(R-L)}$ the critical boundaries for those listening to both stations $s_L = L + \frac{p_R - p_L}{2t(R-L)}$ and $s_R = R + \frac{p_R - p_L}{2t(R-L)}$ and thus the total listening demands facing each station:

$$\begin{aligned} x_L &= \frac{1}{2}\left[\frac{p_R - p_L}{t(R-L)} + (R+L)\right] = s_L + \frac{1}{2}(R-L) \\ x_R &= \frac{1}{2}\left[2 - (R+L) - \frac{p_R - p_L}{t(R-L)}\right] = (1-s_R) + \frac{1}{2}(R-L). \end{aligned}$$

Station $j = L, R$ then chooses p_j to maximise $\pi_j = p_j x_j - F$ which yields the following reaction functions:

$$\begin{aligned} p_L &= 1/2[p_R + t(R+L)(R-L)] \\ p_R &= 1/2\{p_L + t(R-L)[2-(R+L)]\}. \end{aligned}$$

Interestingly, these are identical to those reported above in the standard Hotelling model and, consequently, this version of the subscription model yields exactly the same outcomes as that earlier version. Equilibrium prices, market shares and profits are again:

$$\begin{aligned} p_L &= 1/3t(R-L)[2+(R+L)] \\ p_R &= 1/3t(R-L)[4-(R+L)] \\ x_L &= [2+(R+L)]/6 \\ x_R &= [4-(R+L)]/6 \end{aligned}$$

$$\pi_L = t(R-L)[2+(R+L)]^2/18 - F$$
$$\pi_R = t(R-L)[4-(R+L)]^2/18 - F.$$

In terms of location choices, then, the familiar principle of maximal differentiation still prevails: $L^* = 0$ and $R^* = 1$. So $x_j^* = 1/2$, $P_j^* = t$ and $\pi_j^* = 1/2t - F$. Total welfare is:

$$W = \underline{v} - 2F - \frac{1}{3}t[(x^* - L)^3 + L^3 - (R-1)^3 + (R - x^*)^3]$$

and evaluating this at $L = 0$ and $R = 1$ (so $x^* = 1/2$) gives $W = \underline{v} - 2F - (t/12)$, again in contrast to the level of welfare when locations are chosen to maximise it: at $L = 1/4$ and $R = 3/4$ (so $x^* = 1/2$) we obtain the higher level $W = \underline{v} - 2F - (t/48)$.

First-best Solution

Inspection of (A6) reveals that, when $R > L$, welfare is maximised at $R = 1$ and $L = 0$. Indeed, we can solve directly for the optimal symmetric locations: when $R = 1 - L$ we get $a_L = a_R \equiv a = 2t(1 - 2L)/5$ and we still have $\rho = 1/2$ so we have

$$W = \underline{v} - 2F - \frac{t}{3}[L^3 + L^3] - \frac{4t}{81}(1-2L)(1-1)^2$$
$$= \underline{v} - 2F - \frac{2t}{3}L^3.$$

If we then choose L to maximise welfare, we get:

$$\frac{dW}{dL} = -2tL^2 \leq 0$$

which implies that the optimal symmetric locations are $L = 0$ and $R = 1$. With this maximal differentiation we get higher profits for the stations $(3t/25)$ and greater surplus for advertisers $(2t/25)$. But total consumer welfare is actually lower: even though $U = \underline{v} - a$ (so the only disutility is from advertising) we have greater advertising in equilibrium $(2t/5)$. Indeed, consumer utility is maximised at $L = R = 0.5$ when equilibrium advertising is considered – *minimal* differentiation. To see this, note that in a symmetric equilibrium so $R = 1 - L$, from (13) we have, where $p_L = p_R \equiv p$,

$$a_L(L,R,p_L,p_R) = \frac{1}{3}t(R-L)[2+(R+L) - 2(2p_R + p_L)] = t(1-2L)(1-2p)$$
$$a_R(L,R,p_L,p_R) = \frac{1}{3}t(R-L)[4-(R+L) - 2(2p_R + p_L)] = t(1-2L)(1-2p).$$

But we shall see later that the optimal advertising price, in the symmetric case, is $p = 0.3$ so $a_L = a_R \equiv a = 2t(1 - 2L)/5$ (so $A = a$) and $R = 1 - L$ (so $1 - (R + L) = 0$) so, from (A5):

$$U = \underline{v} - \left[\frac{2t}{5}(1-2L) + \frac{2t}{3}L^3\right]$$
$$\frac{dU}{dL} = \left(\frac{4t}{5} - 2tL^2\right) = \frac{2t}{5}(2 - 5L^2).$$

Subject to the constraint $L \leq R$ (and, therefore, with symmetric locations, that $L \leq 0.5$) this is positive for all values of L. This solution involves zero advertising, note, and every consumer is better off with these locations than in the laissez faire solution which, in turn is better for all than the welfare-maximising solution. Consider a consumer at $s = 0$ (the jointly worst-off consumer in a symmetric equilibrium.) They derive surplus of $\underline{v} - (tL^2 + a)$ in a symmetric equilibrium which, in the laissez-faire case, is $\underline{v} - (0.05^2 + 0.36)t = \underline{v} - 0.3625t$. In the surplus-maximising solution above, however, they receive $\underline{v} - (0.5^2)t \cong \underline{v} - 0.25t$ which is greater. Finally, in the welfare-maximising maximal differentiation solution they get $\underline{v} - a = \underline{v} - 0.4t$.

Derivation of Optimal Laissez-Faire and Collusive Locations

Solving the reaction functions (14) and (15) yields equilibrium prices and these in turn yield equilibrium advertising levels as functions of locations only. We can then use these to get profits as functions of locations only. To summarise:

$$
\begin{aligned}
&p_L(L,R) = \frac{1}{30}[4 + 5(R + L)] \quad p_R(L,R) = \frac{1}{30}[14 - 5(R + L)] \\
&a_L(L,R) = \frac{t(R - L)}{45}[8 + 10(R + L)] \quad a_R(L,R) = \frac{t(R - L)}{45}[28 - 10(R + L)] \\
&\pi_L(L,R) = \frac{t(R - L)}{(45)(15)}[4 + 5(R + L)]^2 - F \quad \pi_R(L,R) = \frac{t(R - L)}{(45)(15)}[14 - 5(R + L)]^2 - F.
\end{aligned}
\tag{A7}
$$

In the first stage of the game, then, the stations choose these locations simultaneously knowing the subsequent prices and advertising levels that will result. Again using firm L as an illustration, it faces the following problem in the first stage:

$$
\begin{aligned}
&\underset{\{L\}}{\text{Max}}\, \pi_L = a_L(L,R)p_L(L,R) - F = \frac{t(R - L)}{(15)(45)}[4 + 5(R + L)]^2 - F \\
&\frac{\partial \pi_L}{\partial L} = 0 \Rightarrow 10(R - L)[4 + 5(R + L)] - [4 + 5(R + L)]^2 = 0 \\
&\Rightarrow 10(R - L) - [4 + 5(R + L)] = 0 \Rightarrow L = \frac{1}{15}(5R - 4).
\end{aligned}
\tag{A8}
$$

A similar exercise for firm R yields:

$$
R = \frac{1}{15}(14 + 5L). \tag{A9}
$$

These two solve for:

$$
\begin{aligned}
L &= -\frac{1}{5}(5R + 4) < 0 \text{ or } L = \frac{1}{15}(5R - 4) \\
R &= \frac{1}{5}(14 - 5L) \text{ or } R = \frac{1}{15}(14 + 5L).
\end{aligned}
$$

Our SOCs require that $\frac{\partial^2 \pi_L}{(\partial L)^2} = -\frac{t}{(15)(45)}(50R + 150L + 80) \leq 0$ which does not hold at $L = -(5R+4)/5$. So $L = (5R-4)/15$ and, similarly, $R = (14+5L)/15$.

To see that the radio stations' optimal collusive locations involve maximal differentiation, note, from (A8), that:

$$\pi_L(L,R) = \frac{t(R-L)}{(45)(15)}[4+5(R+L)]^2 - F \quad \pi_R(L,R) = \frac{t(R-L)}{(45)(15)}[14-5(R+L)]^2 - F$$

so $\pi_L(L, R) + \pi_R(L, R) = \dfrac{t(R-L)}{(45)(15)}(18)^2 - 2F$ which is increasing in $R - L$.

Proof of Proposition Three: If there is an externality attached to local music, first best locations involve $L = 0$ still but $R \leq 1$ with R decreasing in the value of the externality. Advertising at the local music station is zero with advertising at the other station maximised.

From the paper, we have

$$v^R = \underline{v} + \rho M_L - 2F - \frac{t}{3}[L^3 + (1-R)^3] - \frac{4t}{81}(R-L)[1-(R+L)]^2 \tag{18}$$

and

$$M_L = \frac{1}{2}\left[\frac{1}{t}(a_R - a_L) + (R-L)(R+L) + 2(1-R)\right]. \tag{19}$$

Consequently,

(*i*) $\dfrac{\partial v^R}{\partial R} = t(1-R)^2 - \rho(1-R) - \dfrac{4t}{81}[1-(R+L)](1-2R)$

(*ii*) $\dfrac{\partial v^R}{\partial L} = -tL^2 - \rho L + \dfrac{4t}{81}[1-(R+L)](1-2L)$

(*iii*) $\dfrac{\partial v^R}{\partial a_L} = -\dfrac{\rho}{2t} < 0.$

(*iv*) $\dfrac{\partial v^R}{\partial a_R} = \dfrac{\rho}{2t} > 0.$

Condition (*iii*) tells us that advertising at L is optimally reduced to zero and condition (*iv*) that advertising at R should be raised, the final parts of the Proposition. Setting (*ii*) to zero and differentiating yields the first part of the Proposition:

$$\left.\frac{\partial L}{\partial \rho}\right|_{(ii)} = -\frac{\partial(ii)/\partial\rho}{\partial(ii)/\partial L} = \frac{L}{\partial(ii)/\partial L} \leq 0$$

where the sign follows from the SOC for a maximum (and can also be confirmed directly.) We know the optimal value of L when $\rho = 0$ is $L = 0$ so this result confirms that $L = 0$ when $\rho > 0$ too. Finally, setting (*i*) to zero for the optimal location of R gives R as an implicit function of ρ:

$$t(1-R)^2 - \rho(1-R) - \frac{4t}{81}[1-(R+L)](1-2R) = 0.$$

Differentiating,

$$\left.\frac{\partial R}{\partial \rho}\right|_{(i)} = -\frac{\partial(i)/\partial\rho}{\partial(i)/\partial R} = \frac{(1-R)}{\partial(i)/\partial R} \leq 0$$

where the sign follows from the SOC for a maximum. This confirms the second part of the Proposition.

Effects of a Quota

From $R=\gamma$ and $L=0$ we can solve (A7) for the following, using a γ subscript to denote values under a restrictive quota:

$$\begin{aligned}
&L_\gamma = 0 && R_\gamma = \gamma\\
&p_{L\gamma} = \frac{4+5\gamma}{30} && p_{R\gamma} = \frac{14-5\gamma}{30}\\
&a_{L\gamma} = \frac{2t\gamma}{45}(4+5\gamma) && a_{R\gamma} = \frac{2t\gamma}{45}(14-5\gamma) > a_{L\gamma}\\
&\pi_{L\gamma} = \frac{t\gamma}{(45)(15)}(4+5\gamma)^2 - F && \pi_{R\gamma} = \frac{t\gamma}{(45)(15)}(14-5\gamma)^2 - F\\
&x_{L\gamma} = \frac{1}{18}(4+5\gamma) && x_{R\gamma} = \frac{1}{18}(14-5\gamma).
\end{aligned}$$

To see the effect the quota has on the amount of local content played and heard, local content aired is $\frac{1}{2}[(1-L_\gamma)+(1-R_\gamma)]$ or $1-\frac{1}{2}(L_\gamma+\gamma)$ which, with a restrictive quota, is $\frac{1}{2}(2-\gamma)$. But we may have fewer people ($s_{L\gamma}$) listening only to station L and we certainly have a greater fraction $(1-s_{R\gamma})$ listening only to R. For the rest, we need to integrate λ_s over the consumers from $s_{L\gamma}$ to $s_{R\gamma}$, where:

$$s_{L\gamma} = \frac{2}{9}(1-\gamma)(\geq 0.05 \Leftrightarrow \gamma \leq 0.755)$$
$$s_{R\gamma} = s_{L\gamma}+\gamma = \frac{1}{9}(2+7\gamma)(\geq \gamma).$$

Note that $s_{R\gamma}$ always exceeds γ; i.e. there are some consumers who prefer less local content than is provided by R who will still listen to some L. The reason is that there is less advertising on L than on the otherwise preferred R. Total local content heard, then, is:

$$M_{L\gamma} = x_L(1-L)+x_R(1-R) = \frac{1}{18}(5\gamma^2-14\gamma+18).$$

This is greater than 50% for all $\gamma<1$ and $\mathrm{d}M_{L\gamma}/\mathrm{d}\gamma<0$ for all $\gamma<1$ so tightening the quota always leads to more local content being heard overall.

Consider instead the case of a mild quota in which $\gamma\in(0.8, 0.95)$. Again $R=\gamma$ but, solving (A7) with $R=\gamma$, and using m subscripts to denote the mild quota case,

$$L_m = \frac{1}{15}\left(-8-5\gamma+2\sqrt{25\gamma^2+20\gamma+4}\right) = \frac{1}{15}(5\gamma-4).$$

Thus $R_m+L_m = 4/3\left(\gamma-\frac{1}{5}\right)$ and $R_m-L_m = \frac{2}{3}\left(\gamma+\frac{2}{5}\right)$ so we can solve for optimal prices, advertising levels and profits:

$$\begin{aligned}
&L_m = \frac{1}{15}(5\gamma-4) && R_m = \gamma\\
&p_{Lm} = \frac{2}{45}(2+5\gamma) && p_{Rm} = \frac{1}{45}(23-10\gamma)\\
&a_{Lm} = \frac{16t}{(45)^2}(2+5\gamma)^2 && a_{Rm} = \frac{8t}{(45)^2}(2+5\gamma)(23-10\gamma) \geq a_{Lm}
\end{aligned}$$

$$\pi_{Lm} = \frac{32t}{(45)^3}(2+5\gamma)^3 - F \qquad \pi_{Rm} = \frac{8t}{(45)^3}(2+5\gamma)(23-10\gamma)^2 - F$$
$$x_{Lm} = \frac{2}{27}(2+5\gamma) x_{Rm} = \frac{1}{27}(23-10\gamma) \geq x_{Lm}.$$

Derivation of Local Music Heard Under a Quota

With a restrictive quota ($\gamma \leq 0.8$) we have,

$$\begin{aligned} M_{L\gamma} &= x_{L\gamma}(1-L_\gamma) + x_{R\gamma}(1-R_\gamma) \\ &= s_{L\gamma}(1-L_\gamma) + \frac{1}{2}(R_\gamma - L_\gamma)[(1-L_\gamma)+(1-R_\gamma)] + (1-s_{R\gamma})(1-R_\gamma) \\ &= s_{L\gamma} + \frac{1}{2}\gamma[1+(1-\gamma)] + (1-s_{R\gamma})(1-\gamma) \\ &= \frac{2}{9}(1-\gamma) + \frac{\gamma}{2}(2-\gamma) + \frac{7}{9}(1-\gamma)^2 \\ &= \frac{1}{18}(5\gamma^2 - 14\gamma + 18). \end{aligned}$$

Total local music *played*, however, is $\frac{1}{3}[(1-L_\gamma)+(1-R_\gamma)]$ or $1-\frac{1}{2}(L_\gamma+\gamma)$ which, with a restrictive quota, is $\frac{1}{2}(2-\gamma)$.

With a mild quota ($\gamma \in (0.8, 0.95)$) again $R = \gamma$ and we have,

$$\begin{aligned} M_{Lm} &= x_{Lm}(1-L_m) + x_{Rm}(1-R_m) \\ &= s_{Lm}(1-L_m) + \frac{1}{2}(R_m - L_m)[(1-L_m)+(1-R_m)] + (1-s_{Rm})(1-R_m) \\ &= \frac{(2+5\gamma)(19-5\gamma)}{(45)^2} + \frac{(2+5\gamma)[(19-5\gamma)+15(1-\gamma)]}{(15)^2} + \frac{(97-95\gamma)(1-\gamma)}{3(45)} \\ &= \frac{1}{(45)^2}(500\gamma^2 - 1625\gamma + 2105). \end{aligned}$$

Total local music *played* is again $1-\frac{1}{2}(L_m+\gamma)$ which, with a mild quota, is $(17-10\gamma)/15$.

An Alternative Representation of the Effects of a Quota

We can also look at consumption of local content by consumers' locations. From our earlier expressions we can calculate the following:

$$M_L(s) = \left\{\begin{array}{ll} 0.95 & \text{if } s \leq 0.05 = s_L \\ 1-s & \text{if } s \in (s_L, s_R) \\ 0.05 & \text{if } s \geq 0.95 = s_R \end{array}\right\} \text{if there is no quota}$$

$$M_L(s) = \left\{\begin{array}{ll} \frac{1}{15}(19-5\gamma) & \text{if } s \leq \frac{1}{135}(2+5\gamma) = s_{Lm} \\ \frac{1}{135}(173-40\gamma) - s & \text{if } s \in (s_{Lm}, s_{Rm}) \\ 1-\gamma & \text{if } s \geq \frac{19}{135}(2+5\gamma) = s_{Rm} \end{array}\right\} \text{with a mild quota: } \gamma \in (0.8, 0.95)$$

$$M_L(s) = \left\{\begin{array}{ll} 1 & \text{if } s \leq \frac{2}{9}(1-\gamma) = s_{L\gamma} \\ \frac{1}{9}(11-2\gamma) - s & \text{if } s \in (s_{L\gamma}, s_{R\gamma}) \\ 1-\gamma & \text{if } s \geq \frac{1}{9}(2+7\gamma) = s_{R\gamma} \end{array}\right\} \text{with a restrictive quota: } \gamma \leq 0.8$$

We can then plot $M_L(s)$ for different quotas, as illustrated in Figure A1.

So the quota increases demand for local content at every location and though initially it reduces the marginal consumer who listens only to L, this too rises when the quota reaches 20%, exceeding the no-quota level again when the local requirement exceeds 22.5%.

Derivation of Welfare Under a Quota

Consider first a restrictive quota. We have:

$$U = \underline{v} - A - \frac{t}{3}[L^3 + (1-R)^3] - \frac{4t}{81}(R-L)[1-(R+L)]^2.$$

Under the restrictive quota, $L_\gamma = 0$ and $R_\gamma = \gamma$ so

$$\begin{aligned} U_\gamma &= \underline{v} - A_\gamma - \frac{t}{3}(1-\gamma)^3 - \frac{4t\gamma}{81}(1-\gamma)^2 \\ &= \underline{v} - A_\gamma - \frac{t}{81}(27 + 81\gamma^2 - 81\gamma - 27\gamma^3) - \frac{t}{81}(4\gamma + 4\gamma^3 - 8\gamma^2) \\ &= \underline{v} - A_\gamma - \frac{t}{81}(27 - 77\gamma + 73\gamma^2 - 23\gamma^3). \end{aligned}$$

We still have $Y_r = A_\gamma - 2F$ so we have an expression for the regulator's welfare:

$$v_\gamma^R = \underline{v} - 2F + \frac{\rho}{18}(5\gamma^2 - 14\gamma + 18) - \frac{t}{81}(27 - 77\gamma + 73\gamma^2 - 23\gamma^3).$$

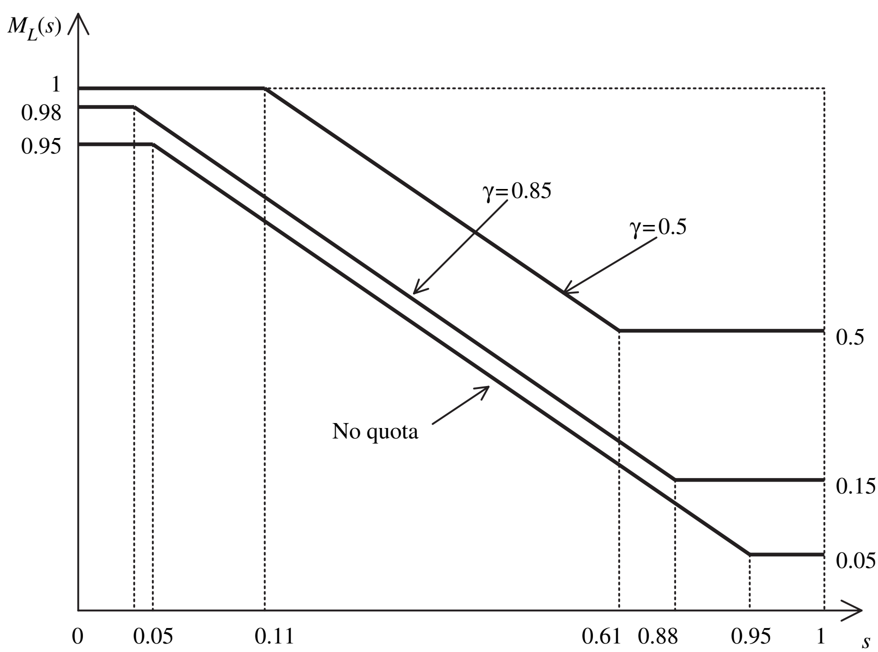


Fig. A1.

From this,

$$\left.\frac{\mathrm{d}v_\gamma^R}{\mathrm{d}\gamma}\right|_{\rho=0} = -\frac{t}{81}(-77 + 146\gamma - 69\gamma^2)$$

$$\left.\frac{\mathrm{d}^2 v_\gamma^R}{(\mathrm{d}\gamma)^2}\right|_{\rho=0} = \frac{t}{81}(138\gamma - 146) < 0 \ \forall\gamma \Rightarrow \left.\frac{\mathrm{d}v_\gamma^R}{\mathrm{d}\gamma}\right|_{\rho=0} \text{ attains a minimum at } \gamma = 0.8$$

$$\left.\frac{\mathrm{d}v_\gamma^R}{\mathrm{d}\gamma}\right|_{\substack{\rho=0\\ \gamma=0.8}} = \frac{109t}{(81)(25)} \cong 0.0538t > 0 \Rightarrow \left.\frac{\mathrm{d}v_\gamma^R}{\mathrm{d}\gamma}\right|_{\rho=0} > 0 \ \forall\gamma.$$

Hence if there is no externality ($\rho = 0$) welfare is increasing in γ for all relevant γ so the quota is always harmful.

Turning to the case of a mild quota and proceeding as before, we have, $L_m = \frac{1}{15}(5\gamma - 4)$ and $R_m = \gamma$ so $R_m - L_m = \gamma - \frac{1}{15}(5\gamma - 4) = \frac{2}{15}(5\gamma + 2)$ and $R_m + L_m = \gamma + \frac{1}{15}(5\gamma - 4) = \frac{4}{15}(5\gamma - 1)$. Thus:

$$\begin{aligned}
U_m &= \underline{v} - A_m - \frac{t}{3}[L_m^3 + (1 - R_m)^3] - \frac{4t}{81}(R_m - L_m)[1 - (R_m + L_m)]^2 \\
&= \underline{v} - A_m - \frac{t}{3}\left[\frac{1}{15^3}(5\gamma - 4)^3 + (1-\gamma)^3\right] - \frac{4t}{81}\left\{\frac{2}{15}(5\gamma + 2)\left[1 - \frac{4}{15}(5\gamma - 1)\right]^2\right\} \\
&= \underline{v} - A_m - \frac{t}{3}\left[\frac{1}{15^3}(5\gamma - 4)^3 + (1-\gamma)^3\right] - \frac{4t}{(81)(15)^3}\{2(5\gamma + 2)[15 - 4(5\gamma - 1)]^2\} \\
&= \underline{v} - A_m - \frac{t}{3(15)^3}(-3,250\gamma^3 + 9,825\gamma^2 - 9,885\gamma + 3,311) \\
&\quad - \frac{4t}{(81)(15)^3}(4,000\gamma^3 - 6,000\gamma^2 + 570\gamma + 1,444) \\
&= \underline{v} - A_m - \frac{t}{(81)(15)^3}(-87,750\gamma^3 + 265,275\gamma^2 - 266,895\gamma + 89,397) \\
&\quad - \frac{t}{(81)(15)^3}(16,000\gamma^3 - 24,000\gamma^2 + 2,280\gamma + 5,776) \\
&= \underline{v} - A_m - \frac{t}{(81)(15)^3}(-71,750\gamma^3 + 241,275\gamma^2 - 264,615\gamma + 95,173).
\end{aligned}$$

But $Y_{\gamma m} = A - 2F$ still and $\rho M_{Lm} = \frac{\rho}{(45)^2}(500\gamma^2 - 1625\gamma + 2105)$ so:

$$\begin{aligned}
v_m^R = \underline{v} - 2F &- \frac{t}{(81)(15)^3}(-71,750\gamma^3 + 241,275\gamma^2 - 264,615\gamma + 95,173) \\
&+ \frac{\rho}{(45)^2}(500\gamma^2 - 1,625\gamma + 2,105).
\end{aligned} \tag{A9}$$

From this,

$$\left.\frac{\mathrm{d}v_m^R}{\mathrm{d}\gamma}\right|_{\rho=0} = \frac{t}{(81)(15)^3}(264,615 - 482,550\gamma + 215,250\gamma^2)$$

$$\left.\frac{\mathrm{d}^2 v_m^R}{(\mathrm{d}\gamma)^2}\right|_{\rho=0} = -\frac{t}{(81)(15)^3}(482,550 + 430,500\gamma) < 0 \ \forall\gamma$$

$$\Rightarrow \left.\frac{\mathrm{d}v_m^R}{\mathrm{d}\gamma}\right|_{\rho=0} \text{ attains a minimum at } \gamma = 0.95$$

$$\left.\frac{\mathrm{d}v_m^R}{\mathrm{d}\gamma}\right|_{\substack{\rho=0\\ \gamma=0.95}} = \frac{455.625t}{(81)(15)^3} = 0.00167t > 0 \Rightarrow \left.\frac{\mathrm{d}v_m^R}{\mathrm{d}\gamma}\right|_{\rho=0} \forall\gamma \in [0.8, 0.95].$$

$$\text{Also, } \left.\frac{\mathrm{d}v_m^R}{\mathrm{d}\gamma}\right|_{\substack{\rho=0\\ \gamma=0.8}} = \frac{16,335t}{(81)(15)^3} \cong 0.05975t.$$

So, again, in the absence of any externality welfare is increasing in γ over the entire range of $\gamma \in [0.8,\ 0.95]$. So welfare again falls as the quota is tightened over this entire range.

Looking at welfare (with $\rho = 0$) over the entire range of the quota yields Figure A2 (not to scale) in which welfare losses are steeper initially until L is driven to play only local music when the quota hits 20%. These curves are derived from the properties calculated above (in particular, at $\gamma = 0.8$ welfare is steeper *in* γ for the mild quota than for the restrictive one.)

If we look at the various components of welfare, however, it is interesting to note that consumers actually gain from a small quota, as illustrated in Figure A3.

In this Figure (also not to scale) consumers' welfare initially rises as the local content requirement is increased from the no-quota level of 5% and then increases more rapidly as it rises beyond $c_1 = 20\%$. It is maximised at around $c_2 \cong 62\%$ and then falls.

Suppose this 'optimal' local content requirement of around 62% were to be imposed. Then station L plays only local music and gets a little under 33% of the market, station R plays around 62% local music and gets a little over 67% of the market and local music is a little under 74% of total music heard. Consumer welfare is approximately $\underline{v}$–0.2567t, compared to the no-intervention level of approximately $\underline{v}$–0.3601t. Station R charges

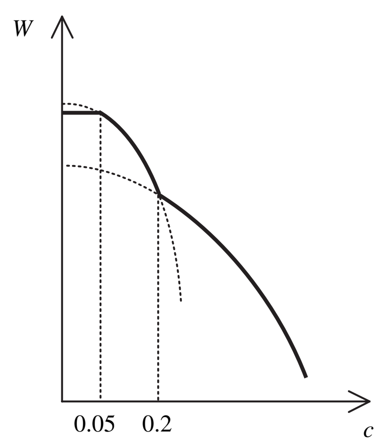


Fig. A2.

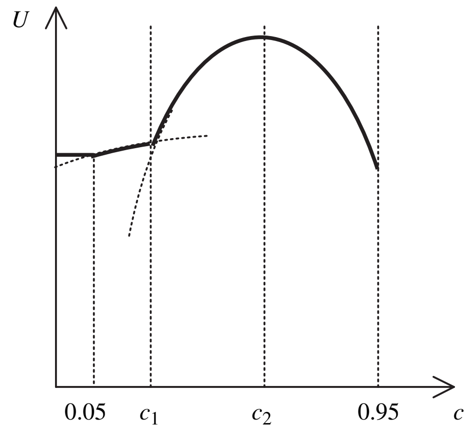


Fig. A3.

more for advertising and places fewer advertisements and makes lower profits than in the absence of a quota, while station L charges less but still places fewer advertisements and so also makes lower profits.

To derive Figure A.3, consider first the mild quota. From

$$U_m = \underline{v} - A_m - \frac{t}{(81)(15)^3}(-71,750\gamma^3 + 241,275\gamma^2 - 264,615\gamma + 95,173)$$

and

$$\begin{aligned} A_m = a_{Lm}x_{Lm} + a_{Rm}x_{Rm} &= \frac{32t}{(27)(45)^2}(2+5\gamma)^3 + \frac{8t}{(27)(45)^2}(2+5\gamma)(23-10\gamma)^2 \\ &= \frac{8t}{(27)(45)^2}(2+5\gamma)[4(2+5\gamma)^2 + (23-10\gamma)^2] \\ &= \frac{8t}{(27)(45)^2}(2+5\gamma)(200\gamma^2 - 380\gamma + 545) \\ &= \frac{8t}{(27)(45)^2}(1000\gamma^3 - 1,500\gamma^2 + 1,965\gamma + 1,090). \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}U_m}{\mathrm{d}\gamma} &= -\frac{\mathrm{d}A_m}{\mathrm{d}\gamma} - \frac{t}{(81)(15)^3}(-215,250\gamma^2 + 482,550\gamma - 264,615) \\ &= -\frac{40t}{(81)(15)^3}(3,000\gamma^2 - 3,000\gamma + 1,965) \\ &\quad - \frac{t}{(81)(15)^3}(-215,250\gamma^2 + 482,550\gamma - 264,615) \\ &= -\frac{t}{(81)(15)^3}(120,000\gamma^2 - 120,000\gamma \\ &\quad + 78,600 - 215,250\gamma^2 + 482,550\gamma - 264,615) \\ &= -\frac{t}{(81)(15)^3}(-95,250\gamma^2 + 362,550\gamma - 186,015) \\ \frac{\mathrm{d}^2U_m}{(\mathrm{d}\gamma)^2} &= -\frac{t}{(81)(15)^3}(-190,500\gamma + 362,550) < 0\ \forall\gamma. \end{aligned}$$

So $\mathrm{d}U_m/\mathrm{d}\gamma$ attains a maximum at $\gamma = 0.8$ at which point

$$\left.\frac{\mathrm{d}U_m}{\mathrm{d}\gamma}\right|_{\gamma=0.8} = -\frac{t}{(81)(15)^3}(-60,960 + 290,040 - 186,015)$$
$$= -\frac{43,065t}{(81)(15)^3} \cong -0.15753t < 0 \Rightarrow \frac{\mathrm{d}U_m}{\mathrm{d}\gamma} < 0 \forall\gamma \in [0.8, 0.95].$$

So U_m is decreasing in γ over the entire range $\gamma \in [0.8, 0.95]$.

Turning to a restrictive quota,

$$A\gamma = a_{L\gamma}x_{L\gamma} + a_{R\gamma}x_{R\gamma} = \frac{2t\gamma}{(18)(45)}(4+5\gamma)^2 + \frac{2t\gamma}{(18)(45)}(14-5\gamma)^2$$
$$= \frac{2t\gamma}{(18)(45)}(16 + 25\gamma^2 + 40\gamma + 196 + 25\gamma^2 - 140\gamma)$$
$$= \frac{2t}{(18)(45)}(50\gamma^3 - 100\gamma^2 + 212\gamma)$$

and from U_γ,

$$\frac{\mathrm{d}U_\gamma}{\mathrm{d}\gamma} = -\frac{\mathrm{d}A_\gamma}{\mathrm{d}\gamma} - \frac{t}{(81)}(-77 + 146\gamma - 69\gamma^2)$$
$$= -\frac{t}{(9)(45)}(150\gamma^2 - 200\gamma + 212) - \frac{t}{(81)}(-77 + 146\gamma - 69\gamma^2)$$
$$= -\frac{t}{(9)(45)}(150\gamma^2 - 200\gamma + 212 - 385 + 730\gamma - 345\gamma^2)$$
$$= \frac{t}{(9)(45)}(173 - 530\gamma + 195\gamma^2)$$
$$\frac{\mathrm{d}^2U_\gamma}{(\mathrm{d}\gamma)^2} = \frac{t}{(9)(45)}(390\gamma - 530) < 0\ \forall\gamma \in [0, 0.8]$$

$$\frac{\mathrm{d}U_\gamma}{\mathrm{d}\gamma} = 0 \Rightarrow 195\gamma^2 - 530\gamma + 173 = 0$$
$$\Rightarrow \gamma = \frac{530 - \sqrt{145,960}}{390} \cong 0.37937.$$

So utility is maximised at around $\gamma = 38\%$ or $c = 1 - \gamma \cong 62\%$. Note also that $\left.\frac{\mathrm{d}U_\gamma}{\mathrm{d}\gamma}\right|_{\gamma=0.8} = -\frac{631t}{(45)^2} \cong -0.3116t$ so the slope of consumer welfare at $\gamma = 0.8$ is steeper for the restrictive quota than for the mild one, as shown in the Figures.

Looking at the other components of welfare, $(\pi_L + \pi_R)$ and $(S_L + S_R)$ are isomorphic and Figure A4 illustrates their sum. While they are monotonically decreasing in the local content requirement, once it exceeds 20% and station L plays only local music, profits and surplus decline more steeply in c although at an initially decreasing rate, as shown.

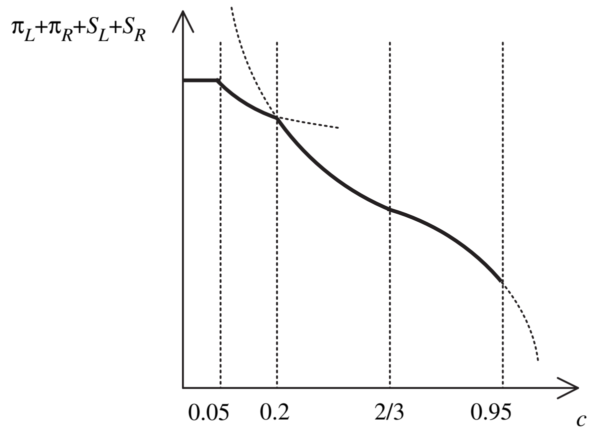


Fig. A4.

To explain Figure A4, consider first a restrictive quota. We have $S_i = (x_i - p_i)a_i$ and $\pi_i = p_{iai} - F$ for $i = L, R$ so:

$$\begin{aligned}
\pi_{L\gamma} + \pi_{R\gamma} &= \frac{t\gamma}{(45)(15)}[(4+5\gamma)^2 + (14-5\gamma)^2] - 2F \\
&= \frac{3}{5}\frac{t}{(9)(45)}(212\gamma - 100\gamma^2 + 50\gamma^3) - 2F \\
S_{L\gamma} &= (x_{L\gamma} - p_{L\gamma})a_{L\gamma} = (4+5\gamma)\left(\frac{1}{18} - \frac{1}{30}\right)(4+5\gamma)\frac{2t\gamma}{45} = \frac{2t\gamma(4+5\gamma)^2}{(45)^2} \\
S_{R\gamma} &= (x_{R\gamma} - p_{R\gamma})a_{R\gamma} = (14-5\gamma)\left(\frac{1}{18} - \frac{1}{30}\right)(14-5\gamma)\frac{2t\gamma}{45} = \frac{2t\gamma(14-5\gamma)^2}{(45)^2} \\
S_{L\gamma} + S_{R\gamma} &= \frac{2t\gamma}{(45)^2}[(4+5\gamma)^2 + (14-5\gamma)^2] \\
&= \frac{2}{5}\frac{t}{(9)(45)}(212\gamma - 100\gamma^2 + 50\gamma^3) = \frac{2}{3}(\pi_{L\gamma} + \pi_{R\gamma}).
\end{aligned} \tag{A8}$$

So:

$$\begin{aligned}
\mathrm{Y} \equiv \pi_L + \pi_R + S_L + S_R &= \frac{t}{(9)(45)}(212\gamma - 100\gamma^2 + 50\gamma^3) - 2F \\
\frac{\mathrm{d}Y}{\mathrm{d}\gamma} &= \frac{t}{(9)(45)}(212 - 200\gamma + 150\gamma^2) > 0 \ \forall\gamma \in [0, 0.8] \\
\frac{\mathrm{d}^2 Y}{(\mathrm{d}\gamma)^2} &= \frac{t}{(9)(45)}(300\gamma - 200\gamma) > (<) 0 \text{ as } \gamma > (<)\frac{2}{3} \\
\left.\frac{\mathrm{d}Y}{\mathrm{d}\gamma}\right|_{\gamma=0.8} &= \frac{t}{(9)(45)}(212 - 160 + 96) = \frac{148t}{(9)(45)} \cong 0.365432t.
\end{aligned}$$

From our expressions for a mild quota we have:

$$\pi_{Lm} + \pi_{Rm} = \frac{8t}{(45)^3}(2+5\gamma)[4(2+5\gamma)^2 + (23-10\gamma)^2] - 2F$$
$$= \frac{8t}{(45)^3}(1090 + 1965\gamma - 1500\gamma^2 + 1000\gamma^3) - 2F$$
$$S_{Lm} = (x_{Lm} - p_{Lm})a_{Lm} = 2(2+5\gamma)\left(\frac{1}{27} - \frac{1}{45}\right)(2+5\gamma)^2 \frac{16t}{(45)^2} = \frac{64t(2+5\gamma)^3}{3(45)^3}$$
$$S_{Rm} = (23-10\gamma)^2\left(\frac{1}{27} - \frac{1}{45}\right)(2+5\gamma)\frac{8t}{(45)^2} = \frac{16t(23-10\gamma)^2(2+5\gamma)}{3(45)^3}$$
$$S_{Lm} + S_{Rm} = \frac{16t(2+5\gamma)}{3(45)^3}[(23-10\gamma)^2 + 4(2+5\gamma)^2]$$
$$= \frac{16t}{3(45)^3}(1090 + 1965\gamma - 1500\gamma^2 + 1000\gamma^3) = \frac{2}{3}(\pi_{Lm} + \pi_{Rm}) \quad (A9)$$

$$Y_m \equiv \pi_{Lm} + \pi_{Rm} + S_{Lm} + S_{Rm} = \frac{40t}{(3)(45)^3}(1{,}090 + 1{,}965\gamma - 1{,}500\gamma^2 + 1{,}000\gamma^3) - 2F$$
$$\frac{dY_m}{d\gamma} = \frac{40t}{(3)(45)^3}(1{,}965 - 3{,}000\gamma + 3{,}000\gamma^2) > 0\ \forall\gamma \in [0.8, 0.95]$$
$$\frac{d^2Y_m}{(d\gamma)^2} = \frac{40t}{(3)(45)^3}(6{,}000\gamma - 3{,}000) > 0 \text{ as } \gamma > \frac{1}{2}$$
$$\left.\frac{dY_m}{d\gamma}\right|_{\gamma=0.8} = \frac{40t}{(3)(45)^3}(1{,}965 - 2{,}400 + 1{,}920) = \frac{1{,}485t}{(3)(45)^3} \cong 0.0054321t.$$

Finally, plotting the value of the externality would simply give us a relationship isomorphic to the heavy line in Figure 2 illustrating M_L.

A Quota in the Subscription Model

From our earlier expressions for the subscription model with usage fees we can derive equilibrium values for all variables of interest when $L = 0$ and $R = \gamma$:

$$\begin{array}{lll} p_L = t\gamma(2+\gamma)/3 & p_R = t\gamma(4-\gamma)/3 & \\ x_L = (2+\gamma)/6 & x_R = (4-\gamma)/6 & \\ \pi_L = t\gamma(2+\gamma)^2/18 & \pi_R = t\gamma(4-\gamma)^2/18 & M_L = 1 + [\gamma(\gamma-4)]/6. \end{array}$$

So local music heard is monotonically increasing in γ and profits of both stations fall as the quota is tightened:

$$\frac{d\pi_L}{d\gamma} = \frac{t}{18}(2+\gamma)(2+3\gamma) > 0$$
$$\frac{d\pi_R}{d\gamma} = \frac{t}{18}(4-\gamma)(4-3\gamma) > 0.$$

This is a consequence of the fact that we have maximal differentiation without a quota here – the quota then elicits no locational response from L.

In terms of welfare, we have $W = \underline{v} - 2F - \frac{1}{3}t(1-\gamma)^3 - (4t/81)\gamma(1-\gamma)^2$ so $\frac{dW}{d\gamma} = \frac{(1-\gamma)t}{81}(77 - 69\gamma) > 0$, i.e. welfare falls monotonically as the quota is tightened, as in our central case. In contrast to that case, however, from

$$
\begin{aligned}
U &= \underline{v} - (p_L x_L + p_R x_R) - \frac{t}{3}[L^3 + (1-R)^3] - \frac{4t}{81}(R-L)[1-(R+L)]^2 \\
&= \underline{v} - \frac{t\gamma}{18}[(2+\gamma)^2 + (4-\gamma)^2] - \frac{t}{3}(1-\gamma)^3 - \frac{4t}{81}\gamma(1-\gamma)^2
\end{aligned}
$$

we have the following:

$$\frac{dU}{d\gamma} = \frac{t}{81}(167 + 96\gamma^2 - 182\gamma) > 0 \forall\gamma \in [0, 1].$$

Thus consumer utility falls monotonically as the quota is tightened, in contrast to our central case.

Simulation with Externality

The model was simulated with the externality to generate a plot of welfare against the value of the externality ρ/t and the content requirement, c. The simulations were performed in GAUSS with $t = 1$ and the program – plot12.prg – is included below.

```
new;
library pgraph;
graphset;
t = 1;
v = 10;
F = 2;
nrho = 30;    @Number of grid points in rho (x axis)@
ngamm = 31;    @Number of grid points in gamma (y axis)@
Wvec = zeros(ngamm,nrho);    @Initialise vectors to fill with data@
gammvec = zeros(1,ngamm);
rhovec = zeros(1,nrho);
gamm = gammst;
gammlim = 0.95;
somminc = (gammlim-gammst)/(ngamm-1);
rho = 0;
rholim = 3 * t/4;
rhoinc = (rholim-rho)/(nrho-1);
rhoiter = 1;
do while rhoiter <= nrho;
gammiter = 1;
rhovec[rhoiter] = rho;
```

```
do while gammiter <= ngamm;
@Free trade welfare, W0@
L = 0.05; R = 0.95;
aR = t * (R-L) * (28 − (10 * (R+L)))/45;
aL = t * (R-L) * (8 + (10 * (R+L)))/45;
xR = (2 − (R+L) − ((aR-aL)/(t * (R-L))))/2;
xL = (R+L + ((aR-aL)/(t * (R-L))))/2;
ML = ((1 − L) * xL) + ((1 − R) * xR);
W0 = v + (rho*ML) − (2 * F) − (t * ((L ∧ 3)
 + ((1 − R) ∧ 3))/3) − (((aR-aL) ∧ 2)/(4 * t * (R-L)));
  @Welfare with a mild quota, Wm@
L = ((5 * gamm) − 4)/15; R=gamm;
aR = t * (R − L) * (28 − (10 * (R+L)))/45;
aL = t * (R − L) * (8 + (10 * (R+L)))/45;
xR = (2 − (R+L) − ((aR-aL)/(t * (R-L))))/2;
xL = (R + L + ((aR-aL)/(t * (R-L))))/2;
ML = ((1 − L) * xL) + ((1 − R) * xR);
Wm = v + (rho*ML) − (2 * F) − (t * ((L ∧ 3)
+ ((1 − R) ∧ 3))/3) − (((aR-aL) ∧ 2)/(4 * t * (R-L)));
@Welfare with a restrictive quota, Wr@
L = 0; R = gamm;
aR = t*(R-L) * (28 − (10 * (R+L)))/45;
xR = (2 − (R+L) − ((aR-aL)/(t*(R-L))))/2;
xL = (R+L+((aR-aL)/(t*(R-L))))/2;
ML = ((1 − L) * xL) + ((1 − R) * xR);
Wr = v + (rho*ML) − (2 * F) − (t * ((L ∧ 3) + ((1 − R) ∧ 3))/3)
gammvec[gammiter] = 1 − gamm;
if gamm lt0.8; Wvec[gammiter,rhoiter] = Wr;
else; Wvec[gammiter,rhoiter] = Wm;
endif;
gammiter = gammiter+1;
gamm = gamm+gamminc;
endo;
rhoiter = rhoiter+1;
rho = rho+rhoinc;
gamm = gammst;
endo;
```

```
@The following then plots this lot. The first instruction denotes the output file@
_ptek = "c : gaussprgcultureWelfare.tkf";
_pdate = "";    @The next thing suppresses the date in printing@
gammvec = gammvec';
gammvec = rev(gammvec);
gammvec = gammvec';
Wvec = rev(Wvec);
print "c=" gammvec;
_plwidth = 5;
graphprt(" - c = 1 - cf = welfare.eps -co = l - cs=f - cm=1, 1, 1, 1");
ylabel("Content requirement, c");
xlabel("Externality value, rho/t");
zlabel("Welfare");
surface(rhovec,gammvec',Wvec);
end;
```

Derivations of Results for a Ceiling on Advertising

Suppose the amount of advertising each station can place is limited to a. Suppose, initially, that this ceiling is 'very low' in a sense to be made clear below. Then it will constrain both stations and each will simply choose its price such that the demand from advertisers is exactly a. Inverting (13) gives:

$$\begin{aligned} p_L(L,R,\underline{a},p_R) &= \frac{1}{4}\left[2+(R+L)-2p_R-\frac{3\underline{a}}{t(R-L)}\right] \\ p_R(L,R,\underline{a},p_L) &= \frac{1}{4}\left[4-(R+L)-2p_L-\frac{3\underline{a}}{t(R-L)}\right] \end{aligned} \tag{A10}$$

and solving these reaction functions gives:

$$\begin{aligned} p_L(L,R,\underline{a}) &= \frac{1}{2}\left[(R+L)-\frac{\underline{a}}{t(R-L)}\right] \\ p_R(L,R,\underline{a}) &= 1-\frac{1}{2}\left[(R+L)+\frac{\underline{a}}{t(R-L)}\right]. \end{aligned} \tag{A11}$$

In the first stage of the game, then, radio station $j = L, R$ chooses its location to maximise $\pi_j = p_j\underline{a}$. From (A11) and assuming a symmetric solution we can solve for equilibrium locations and hence prices and profits as functions of the limit, $\underline{a}$:

$$\begin{aligned} L &= \tfrac{1}{2}-\sqrt{\frac{\underline{a}}{4t}} & R &= \tfrac{1}{2}+\sqrt{\frac{\underline{a}}{4t}} \\ p_L = p_R &= \tfrac{1}{2}\left(1-\sqrt{\frac{\underline{a}}{t}}\right) & \pi_L = \pi_R &= \frac{\underline{a}}{2}\left(1-\sqrt{\frac{\underline{a}}{t}}\right)-F \\ x_L &= \frac{1}{2} & x_R &= \frac{1}{2}. \end{aligned} \tag{A12}$$

As noted earlier, the equilibrium we describe above applies only when the advertising limit is severe. To see this, note that in the absence of any constraint, from (16), any symmetric solution has $a_L = a_R \equiv a = 2(1-2L)t/5$ compared to $\underline{a} = 4t(1/2-L)^2$ here, from (A12). So, given locations, the constraint only binds if $\underline{a} \leq a$ which can be rewritten as $L \in [0.3, 0.5]$ which, in turn, requires $\underline{a} \in [0, 0.16t]$ from (A12). Thus our analysis above only holds if the advertising ceiling is less than $4t/25$. Suppose, instead, that $\underline{a} \in [0.16t, 0.36t]$ where the upper limit is the laissez-faire level of advertising. From the expressions for equilibrium advertising in (16), and assuming a symmetric solution, we can solve for:

$$L = \frac{1}{2}\left(1 - \frac{5}{2}\frac{\underline{a}}{t}\right) \equiv L' \text{ and } R = \frac{1}{2}\left(1 + \frac{5}{2}\frac{\underline{a}}{t}\right) \equiv R'.$$

Now, suppose these locations *are* chosen. Would either station then wish to move, given the advertising limit? Note first that

$$\frac{\partial a_L}{\partial L} = \frac{t}{45}[10(R-L) - 8 - 10(R+L)] = -\frac{4t}{45}(2+5L) < 0$$
$$\frac{\partial a_R}{\partial L} = \frac{t}{45}\{-[28 - 10(R+L)] - 10(R-L)\} = -\frac{4t}{45}(7-5L) < 0.$$

So L could be *increased* without violating the constraint, but not decreased. Anyway,

$$\frac{\mathrm{d}\pi_L}{\mathrm{d}L} = \frac{t}{(15)(45)}\{10[4+5(R+L)](R-L) - [4+5(R+L)]^2\} = \frac{t[4+5(R+L)]}{(15)(45)}(5R-15L-4).$$

Evaluating this at the locations calculated above for L and R, yields:

$$\left.\frac{\mathrm{d}\pi_L}{\mathrm{d}L}\right|_{\substack{L=L'\\R=R'}} = \frac{t[4+5(R+L)]}{(15)(45)}\left(\frac{5}{2} + \frac{25}{4}\frac{\underline{a}}{t} - \frac{15}{2} + \frac{25}{4}\frac{\underline{a}}{t} - 4\right) = \frac{t[4+5(R+L)]}{(15)(45)}\left(\frac{25}{2}\frac{\underline{a}}{t} - 9\right) < 0$$

where the negative sign follows from the fact that $\underline{a} \leq 0.36t$. So, ignoring the constraint, L would like to move down the spectrum, away from R at these locations. Doing so, however, will run into the advertising constraint. If L is reduced, given R, then p_L must be increased so that $a_L = \underline{a}$ still. From (A11) we have the price that must be set and solving L's problem then gives,

$$\frac{\mathrm{d}\pi_L}{\mathrm{d}L} = \frac{\underline{a}}{2t(R-L)^2}[t(R-L)^2 - \underline{a}]$$

$$\left.\frac{\mathrm{d}\pi_L}{\mathrm{d}L}\right|_{\substack{L=L'\\R=R'}} = \frac{\underline{a}}{2t(R-L)^2}\left(\frac{25t}{4}\frac{\underline{a}^2}{t^2} - \underline{a}\right) = \frac{\underline{a}}{2t(R-L)^2}\left[\frac{\underline{a}}{4t}(25\underline{a} - 4t)\right] = \frac{\underline{a}^2(25\underline{a} - 4t)}{8t^2(R-L)^2} > 0$$

where the sign follows because $\underline{a} \geq 0.16t$. So, given the constraint, L does not wish to move down the spectrum either. Thus $L = L'$ is a best response to $R = R'$ and a similar exercise shows the same for R'.

Turning to welfare, we again get:

$$W = \pi_L + \pi_R + U + S_L + S_R + \rho M_L$$

where

$$U = \underline{v} - \{\int_0^L [a + t(L-s)^2]\mathrm{d}s + \int_L^R (a)\mathrm{d}s + \int_R^1 [a + t(s-R)^2]\mathrm{d}s\}$$
$$= \underline{v} - \left\{\left(aL + \frac{t}{3}L^3\right) + a(R-L) + \left[a(1-R) + \frac{t}{3}(1-R)^3\right]\right\}.$$

Now, when the advertising ceiling is very low ($\underline{a} \in [0,0.16t]$) denoted with L superscripts, we get:

$$U^L = \underline{v} - \left[2\left(\underline{a}L + \frac{t}{3}L^3\right) + \frac{1}{2}\underline{a}\sqrt{\frac{\underline{a}}{t}}\right] = \underline{v} - \frac{5}{4}\underline{a} - \frac{1}{12}\left(t - \sqrt{\frac{\underline{a}^3}{t}} - 3\sqrt{\underline{a}t}\right).$$

But:

$$\pi_L^L + \pi_R^L = \underline{a}\left(1 - \sqrt{\frac{\underline{a}}{t}}\right) - 2F \rho M_L^L = \frac{\rho}{2} \text{ and } S_L^L + S_R^L = \underline{a}\sqrt{\frac{\underline{a}}{t}}$$
$$\text{so } \pi_L^L + \pi_R^L + S_L^L + S_R^L = \underline{a} - 2F$$

and:

$$W^L = \underline{v} + \frac{\rho}{2} - \frac{1}{4}\underline{a} - \frac{1}{12}\left(t - \sqrt{\frac{\underline{a}^3}{t}} - 3\sqrt{\underline{a}t}\right) - 2F. \tag{A13}$$

But when the ceiling is less restrictive ($\underline{a} \in [0.16t,0.36t]$) denoted with H superscripts, we have

$$U^H = \underline{v} - \left[\underline{a} - \frac{5}{2}\frac{\underline{a}^2}{t} + \frac{t}{12}\left(1 - \frac{5}{2}\frac{\underline{a}}{t}\right)^3 + \frac{5}{2}\frac{\underline{a}^2}{t}\right] = \underline{v} - \left(\frac{t}{12} + \frac{3}{8}\underline{a} + \frac{75}{48t}\underline{a}^2 - \frac{125}{96t^2}\underline{a}^3\right).$$

But:

$$\pi_L^H + \pi_R^H = \frac{3\underline{a}}{5} - 2F, \rho M_L^H = \frac{\rho}{2}, S_L^H + S_R^H = \frac{2}{5}\underline{a} \Rightarrow \pi_L^H + \pi_R^H + S_L^H + S_R^H = \underline{a} - 2F$$

and:

$$W^H = \underline{v} + \frac{\rho}{2} + \left(\frac{5}{8}\underline{a} - \frac{75}{48t}\underline{a}^2 + \frac{125}{96t^2}\underline{a}^3\right) - \frac{t}{12}. \tag{A14}$$

From this we get:

$$\frac{\mathrm{d}W^L}{\mathrm{d}\underline{a}} = \frac{1}{8}\left(\sqrt{\frac{\underline{a}}{t}} + \sqrt{\frac{t}{\underline{a}}} - 2\right) = \frac{1}{8\sqrt{\underline{a}t}}(\underline{a} + t - 2\sqrt{\underline{a}t}) > 0 \ \forall \underline{a} < \frac{4}{25}t$$

$$\frac{\mathrm{d}W^H}{\mathrm{d}\underline{a}} = \frac{5}{8} - \frac{75}{24t}\underline{a} + \frac{125}{32t^2}\underline{a}^2 > 0 \ \forall \underline{a} \in \left[\frac{4t}{25}, \frac{9t}{25}\right].$$

The sign of the latter term follows because:

$$\frac{\mathrm{d}^2 W^H}{(\mathrm{d}\underline{a})^2} = \frac{125}{16t^2}\underline{a} - \frac{25}{8t} = \frac{25}{8t}\left(\frac{5\underline{a}}{2t} - 1\right) < 0 \ \forall \underline{a} < \frac{2t}{5}.$$

But $\left.\frac{\mathrm{d}W^H}{\mathrm{d}\underline{a}}\right|_{\underline{a}=2t/5} = \frac{5}{8} - \frac{75}{24t}\frac{2t}{5} + \frac{125}{32t^2}\frac{4t^2}{25} = \frac{5}{8} - \frac{15}{12} + \frac{5}{8} = 0.$

Thus, welfare is everywhere decreasing as advertising is restricted. Looking at consumer welfare alone, however:

$$\begin{aligned}\frac{\mathrm{d}U^L}{\mathrm{d}\underline{a}} &= \frac{1}{8\sqrt{\underline{a}t}}(\underline{a}+t) - \frac{5}{4}\\ \frac{\mathrm{d}^2U^L}{(\mathrm{d}\underline{a})^2} &= \frac{1}{16\sqrt{\underline{a}^3t}}(\underline{a}-t) < 0 \ \forall \underline{a} < \frac{4t}{25}.\end{aligned} \tag{A15}$$

Setting $\mathrm{d}U^L/\mathrm{d}\underline{a} = 0$, we can show that consumer welfare is maximised – in the low ceiling range $\underline{a} \leq 0.16t$ – at $\underline{a} = \frac{1}{2}[98t \pm \sqrt{(98t)^2 - 4t^2}]$ or $\underline{a} \cong 0.01t$ at which point we get $U \cong \underline{v} - 0.0707t$. And

$$\frac{\mathrm{d}U^H}{\mathrm{d}\underline{a}} = \frac{125}{32t^2}\underline{a}^2 - \frac{75}{24t}\underline{a} - \frac{3}{8} \text{ and } \frac{\mathrm{d}^2U^H}{(\mathrm{d}\underline{a})^2} = \frac{125}{16t^2}\underline{a} - \frac{75}{24t} \leq 0 \ \forall \underline{a} \in \left[\frac{4t}{25}, \frac{2t}{5}\right]$$

$$\left.\frac{\mathrm{d}U^H}{\mathrm{d}\underline{a}}\right|_{\underline{a}=0.16t} = \frac{125}{32t^2}\frac{16t^2}{25^2} - \frac{75}{24t}\frac{4t}{25} - \frac{3}{8} = \frac{1}{10} - \frac{1}{2} - \frac{3}{8} = -\frac{31}{40} < 0.$$

That is, the cubic function U^H is decreasing over the entire relevant range of $\underline{a}$ and thus reaches a maximum at $\underline{a} = 0.16$. So our earlier calculation holds: consumer welfare is maximised in the low ceiling range at $\underline{a} \cong 0.01$ giving consumer surplus of $U \cong \underline{v} - 0.0707t$ which compares favourably to the laissez-faire level of consumer welfare, $(\underline{v} - 0.3601t)$.

Calculations for a Public Station

We look for an equilibrium in which $P < L < R$. There are a number of possible listening patterns that might arise for consumers. Some might listen to only P, only L or only R, others might listen to both P and L or to both L and R, as before, but it is also possible that some might listen to both P and R: the lack of advertising at P makes it attractive. But any equilibrium must involve $a_L < a_R$. If not then no consumers would listen to L at all: a consumer between R and P can create their own ideal content mix by combining those two stations with lower advertising than if they listened to any L. Indeed, for any locations $0 = P < L < R$ it must be the case that a_L is such that no consumer mixes P and R. The reason is that if any consumer finds it optimal to combine P and R then all consumers between P and R must do so too, thus leaving no market for L. This is most easily seen in Figure A5. Suppose locations and advertising are such that consumers' surplus from station P, L and R, as a function of location s, is as shown by $U_P(s)$, $U_L(s)$ and $U_R(s)$ respectively. As shown, any consumer in the interval $P + R$ does better by combining P and R rather than consuming any L at all.

So there are only two possible configurations of advertising and audience mixes here, given locations. One is that a_L is sufficiently low that we have a situation where consumers listen to only P, only L, only R, both P and L or to both L and R. It can be shown that such an outcome cannot be an equilibrium; instead, station L will increase its advertising to the point at which consumers are just indifferent between consuming P and L and consuming L and R. Thus $a_L R = a_R L$. This reduces its total market share but increases its advertising and is illustrated in Figure A6.

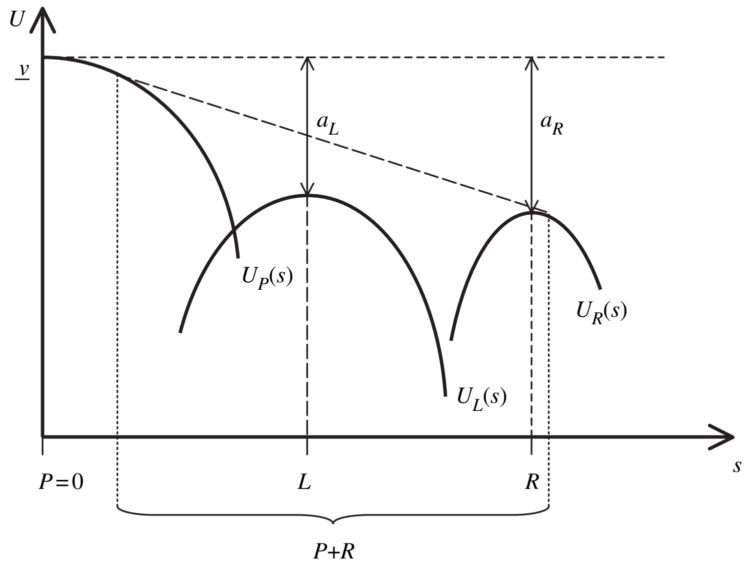


Fig. A5.

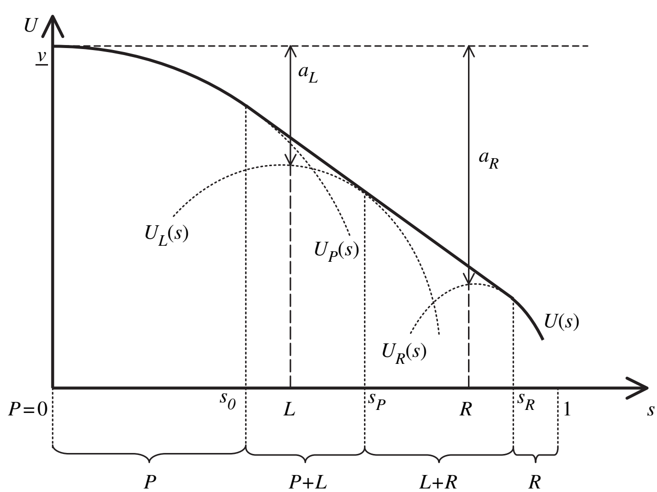


Fig. A6.

With a public station and $x_L = 0$, there is a possibility, of a nonexistence problem here.[1] Looking at Figure A7, we have $a_L(p_L, x_L)R = a_R(p_R, x_R)L$. Let $x_R' = x_R + x_L$. Now suppose that, given R and L, station R instead sets $p_R' >> p_R$ to yield $a_R'(p_R', x_R') = a_R - \varepsilon < a_R$ such that no consumers listen to L at all (see Figure A6.) Then R's market

[1] This is rather different to the non-existence problems (for pure strategies) that can plague standard Hotelling-type models. See Osborne and Pitchik (1987) for a discussion.

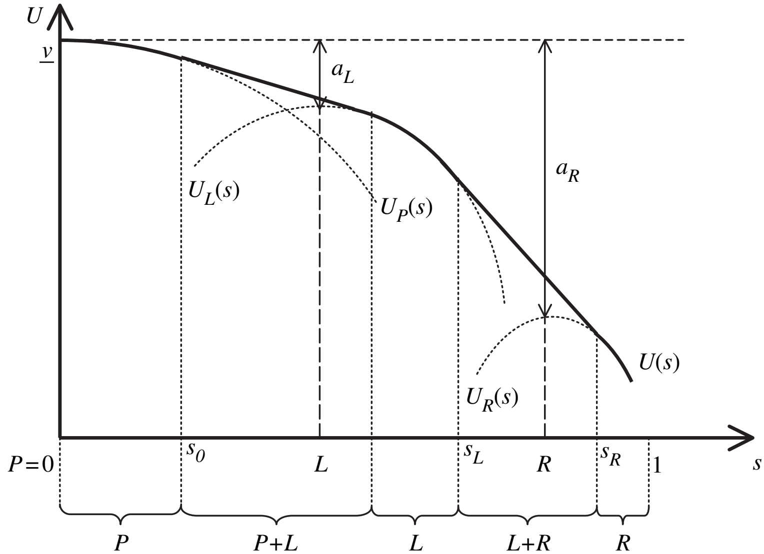


Fig. A7.

share will be approximately x_R' and because $a_R' = a_R(p_R', x_R') \cong a_R(p_R, x_R)$ its profits would seem to be discretely higher than with price p_R, suggesting that the latter cannot be part of a Nash equilibrium. Edgeworth paradox-like, this reasoning would suggest no pure equilibrium exists in prices for any locations as R's profit function is discontinuous in its price. However, this argument ignores the fact that demand for advertising at L also depends on p_R. If firm R were to choose its price to 'undercut' L's advertising levels, the fall in L's market share would also decrease demand for advertising at L, given p_L. Thus a small rise in p_R does not lead to a discrete increase in π_R but rather changes π_R by trading off a higher price against lower advertising, a trade-off that has approximately zero effect on profits when p_R is chosen optimally.

We argue in the article that there are only two possible configurations of advertising and audience mixes here, given locations. One is that a_L is sufficiently low that we have the situation in Figure A7 where consumers listen to only P, only L, only R, both P and L or to both L and R.

Alternatively, station L might increase its advertising to the point at which consumers are just indifferent between consuming P and L and consuming L and R. This reduces its total market share but increases its advertising and is illustrated in Figure A7.

Proceeding as before, consider first an equilibrium such as that shown in Figure A6. We first need to identify the critical locations at the boundaries of each market segment. For a consumer listening to both P and L we have:

$$\begin{aligned}
\Gamma_{PL} &= \lambda_{PL}P + (1-\lambda_{PL})L = (1-\lambda_{PL})L \\
&\Rightarrow U_{PL} = \underline{v} - t[(1-\lambda_{PL})L - s]^2 - (1-\lambda_{PL})a_L \\
&\Rightarrow \frac{\mathrm{d}U_{PL}}{\mathrm{d}\lambda_{PL}} = 2tL[(1-\lambda_{PL})L - s] + a_L = 0 \\
&\Rightarrow \lambda_{PL} = \frac{1}{2tL^2}(2tL^2 + a_L - 2tLs).
\end{aligned}$$

So a consumer will listen only to station P if $\lambda_{PL} \geq 1$ or $s \leq s_0$ where:

$$s_0 \equiv \frac{a_L}{2tL}. \tag{A16}$$

Likewise, a consumer will listen only to station L if $\lambda_{PL} \leq 0$ or $s \geq s_P$ where:

$$s_P \equiv L + \frac{a_L}{2tL} > L. \tag{A17}$$

Turning instead to a consumer listening to both R and L we have:

$$\begin{aligned}
\Gamma_{RL} &= \lambda_{RL}L + (1 - \lambda_{RL})R \\
&\Rightarrow U_{RL} = \underline{v} - t[\lambda_{RL}L + (1 - \lambda_{RL})R - s]^2 - [\lambda_{RL}a_L + (1 - \lambda_{RL})a_R] \\
&\Rightarrow \frac{\mathrm{d}U_{RL}}{\mathrm{d}\lambda_{RL}} = 0 \\
&\Rightarrow \lambda_{RL} = \frac{R - s}{R - L} + \frac{a_R - a_L}{2t(R - L)^2}.
\end{aligned}$$

So a consumer will listen only to station L if $\lambda_{RL} \geq 1$ or $s \leq s_L$ where:

$$s_L \equiv L + \frac{(a_R - a_L)}{2t(R - L)}. \tag{A18}$$

Likewise, a consumer will listen only to station R if $\lambda_{RL} \leq 0$ or $s \geq s_R$ where:

$$s_R \equiv R + \frac{(a_R - a_L)}{2t(R - L)} > R. \tag{A19}$$

We can derive the market shares of the stations as before but with a third station, P:

$$\begin{aligned}
x_P &= \int_0^{s_0} f(s)\mathrm{d}s + \int_{s_0}^{s_P} \lambda_{PL} f(s)\mathrm{d}s \\
&= s_0 + \frac{1}{2}(s_p - s_0) = \frac{1}{2}(s_p + s_0) \\
&= \frac{1}{2tL}(a_L + tL^2).
\end{aligned} \tag{A20}$$

$$\begin{aligned}
x_L &= \int_{s_0}^{s_P} (1 - \lambda_{PL}) f(s)\mathrm{d}s + \int_{s_P}^{s_L} f(s)\mathrm{d}s + \int_{s_L}^{s_R} \lambda_{RL} f(s)\mathrm{d}s \\
&= \frac{1}{2}(s_p - s_0) + (s_L - s_P) + \frac{1}{2}(s_R - s_L) = \frac{1}{2}(s_L + s_R - s_p - s_0) \\
&= \frac{1}{2}\left[R + \frac{(a_R - a_L)}{t(R - L)} - \frac{a_L}{tL}\right].
\end{aligned} \tag{A21}$$

Note that the expression for x_L applies only if $s_L \geq s_P$. Otherwise there are no consumers who listen only to station L and we are in the setting of Figure A5.

$$\begin{aligned} x_R &= \int_{s_L}^{s_R} (1-\lambda_{RL}) f(s)\mathrm{d}s + \int_{s_R}^{1} f(s)\mathrm{d}s \\ &= \frac{1}{2}(s_R - s_L) + (1 - s_R) = \frac{1}{2}(2 - s_R - s_L) \\ &= \frac{1}{2}\left[2 - (R+L) - \frac{(a_R - a_L)}{t(R-L)}\right]. \end{aligned} \tag{A22}$$

From these expressions we can calculate total advertising exposure at the two commercial stations, $a_i x_i$ for $i = L, R$, and maximise advertisers' surplus over the choice of a_i given prices and locations:

$$\begin{aligned} p_L &= \frac{\mathrm{d}(a_L x_L)}{\mathrm{d}a_L} = \frac{1}{2}\left[R + \frac{a_R}{t(R-L)} - \frac{2Ra_L}{tL(R-L)}\right] \\ &\Rightarrow a_L = \frac{1}{2R}[a_R L + tL(R-L)(R-2p_L)] \\ p_R &= \frac{\mathrm{d}(a_R x_R)}{\mathrm{d}a_R} = \frac{1}{2}\left[2 - (R+L) + \frac{a_L}{t(R-L)} - \frac{2a_R}{t(R-L)}\right] \\ &\Rightarrow a_R = \frac{1}{2}[a_L + 2tL(R-L)(1-p_R) - t(R-L)(R+L)]. \end{aligned} \tag{A23}$$

Solving these two jointly gives:

$$\begin{aligned} a_L(p_L, p_R, R, L) &= \frac{tL(R-L)}{4R-L}[2 + (R-L) - 2p_R - 4p_L] \\ a_R(p_L, p_R, R, L) &= \frac{t(R-L)}{4R-L}(4R - 2R^2 - RL - 4Rp_R - 2Lp_L). \end{aligned} \tag{A24}$$

The commercial stations then seek to set advertising rates to maximise their profits i.e. to choose p_i to maximise $\pi_i = a_i p_i - F$ where $i = L, R$. First order conditions for these problems are:

$$\begin{aligned} p_L &= \frac{1}{8}[2 + (R-L) - 2p_R] \\ p_R &= \frac{1}{8}\left[4 - R - (R-L) - \frac{2L}{R}p_L\right] \end{aligned} \tag{A25}$$

which solve jointly for equilibrium prices as functions of location:

$$\begin{aligned} p_L(L,R) &= \frac{R(4 + 6R - 3L)}{2(16R - L)} \\ p_R(L,R) &= \frac{(16R - 8R^2 - 5LR - 2L + L^2)}{2(16R - L)}. \end{aligned} \tag{A26}$$

Substituting into our expressions for a_L and a_R we can get expressions for advertising levels as functions purely of locations:

$$\begin{aligned} a_L(L,R) &= \frac{2tLR(R-L)(4 + 6R - 3L)}{(4R-L)(16R-L)} \\ a_R(L,R) &= \frac{2tR(R-L)(16R - 8R^2 - 5LR - 2L + L^2)}{(4R-L)(16R-L)}. \end{aligned} \tag{A27}$$

From these expressions we can write out the stations' profits as functions of location – $\pi_i(L, R) = p_i(L, R)a_i(L, R) - F$ – and derive first-order conditions for optimal locations. These are very messy functions of R and L but they can be solved numerically. Doing so yields the following solutions:

$$\begin{aligned} L &\cong 0.3856 \qquad R \cong 0.8161 \\ \Rightarrow a_R &\cong 0.1066 \quad a_L \cong 0.0543. \end{aligned} \tag{A28}$$

Now, we need to confirm that in fact there are consumers who listen only to station L; that is, that $s_L \geq s_P$. From our expressions for these critical boundaries, this is equivalent to requiring that $a_L R \geq a_R L$ which is violated by the solutions given above. Thus there are no consumers listening to L at all.

Accordingly, L would reduce its advertising until consumers are (approximately) indifferent to listening to L with another station as to listening to R and P. That is, we have $a_L R = a_R L$. So while the expressions derived earlier for audience shares of R and P are still valid, the share of L becomes:

$$x_L = \int_{s_0}^{s_p} \lambda_{PL} f(s)\mathrm{d}s + \int_{s_p}^{s_R} \lambda_{RL} f(s)\mathrm{d}s = \frac{1}{2}(s_R - s_0) = \frac{1}{2}\left[R + \frac{a_R - a_L}{2t(R-L)} - \frac{a_L}{2tL}\right].$$

Thus

$$\begin{aligned} p_L &= \frac{\mathrm{d}(a_L x_L)}{\mathrm{d}a_L} = \frac{1}{2}\left[R + \frac{a_R - 2a_L}{2t(R-L)} - \frac{a_L}{tL}\right] \Rightarrow \\ a_L &= \frac{L}{R}t(R-L)\left[R + \frac{a_R}{2t(R-L)} - 2p_L\right]. \end{aligned} \tag{A29}$$

Now, we need the price at L to be such that $Ra_L = La_R$ so, from (A29), this requires:

$$p_L = \frac{1}{2}\left[R - \frac{a_R}{2t(R-L)}\right].$$

For station R the problem it faces is the same as in the two-station case (since its only direct rival is its neighbour L). Thus, from (13) and (16),

$$p_R = \frac{1}{8}\left[4 - (2R + L) + \frac{a_R}{t(R-L)}\right]$$

and

$$a_R = \frac{1}{3}t(R-L)[4 - (R+L) - 2(2p_R + p_L)].$$

From our expressions for p_R and p_L we can solve for a_R as follows:

$$a_R(L, R) = \frac{1}{5}t(R-L)[4 - (2R + L)]. \tag{A30}$$

Thus:

$$p_R = \frac{3}{20}[4 - (2R + L)].$$

and:

$$\pi_R(L,R) = p_R(L,R)a_R(L,R) - F = \frac{3}{100}t(R-L)[4-(2R+L)]^2 - F. \qquad (A31)$$

Similarly, from (A30) we can rewrite a_L and p_L and thus derive $\pi_L = a_L p_L - F$:

$$p_L(L,R) = \frac{1}{2}\left\{R - \frac{1}{10}[4-(2R+L)]\right\}$$

$$a_L(L,R) = \frac{L}{R}a_R = \frac{tL(R-L)}{5R}[4-(2R+L)]$$

$$\pi_L(L,R) = \frac{L}{10R}t(R-L)[4-(2R+L)]\left\{R - \frac{1}{10}[4-(2R+L)]\right\} - F.$$

Maximising π_R and π_L over R and L respectively yields the following first order conditions, which solve for L and R:

$$\begin{aligned} \{12R - [4-(2R+L)]\}[(R-2L)\{.\} - L(R-L)]^{+}L(R-L)\{.\} = 0 \Rightarrow L \\ 4 - 6R + 3L = 0 \Rightarrow R. \end{aligned} \qquad (A32)$$

These yield the following feasible solution:

$$L = \frac{2\sqrt{489}-10}{87} \cong 0.3934 \quad R = \frac{318+6\sqrt{489}}{522} \cong 0.8634. \qquad (21)$$

From this we can calculate all the variables of interest:

$$\begin{array}{ll} & \pi_P* = -F \\ p_L* \cong 0.3534 & p_R* \cong 0.2741 \\ a_L* \cong 0.0671t & a_R* \cong 0.1472t \geq a_L* \\ \pi_L* \cong 0.0237t - F & \pi_R* \cong 0.0404t - F \\ x_L* \cong 43\% & x_R* \cong 29\% \\ x_P* \cong 28\%. & \end{array} \qquad (A33)$$

Local music *played* is $1/3[1 + (1-L) + (1-R)]$ or a little over 58.1% of total music played. Local music *heard*, however, is $x_P + x_L(1-L) + x_R(1-R)$ or a little under 58.3% of total music heard.

Turning to welfare, we now have some consumers listening only to P and getting welfare of:

$$\begin{aligned} U_P &= \int_0^{s_0} (\underline{v} - ts^2)f(s)\mathrm{d}s \\ &= \underline{v}s_0 - s_0^3\left(\frac{t}{3}\right) \\ &= \underline{v}s_0 - \frac{1}{3t^2}\left(\frac{a_L}{2L}\right)^3. \end{aligned} \qquad (A34)$$

There are consumers listening to both P and L and getting surplus of:

$$
\begin{aligned}
U_{PL} &= \int_{s_0}^{s_P} \{\underline{v} - t[(1-\lambda_{PL})L - s]^2 - (1-\lambda_{PL})a_L\} f(s)\mathrm{d}s \\
&= \underline{v}(s_P - s_0) + \frac{1}{t}\left(\frac{a_L}{2L}\right)^2 (s_P - s_0) - \frac{a_L}{2L}(s_P^2 - s_0^2) \qquad (A35)\\
&= \underline{v}(s_P - s_0) - \frac{1}{2}a_L(s_P - s_0) - \frac{a_L}{4tL}.
\end{aligned}
$$

which follows from our earlier expressions for s_P, s_0 and λ_{PL}. Turning to consumers listening to both L and R and using the earlier expressions for s_P, s_R and λ_{RL}, we get:

$$
1 - \lambda_{RL} = \frac{1}{2t(R-L)^2}[2t(R-L)(s-L) - (a_R - a_L)]
$$

$$
\Rightarrow \lambda_{RL}L + (1-\lambda_{RL})R - s =
$$

$$
\frac{[2tR(R-L)(s-L) - R(a_R - a_L) + 2tL(R-L)(R-s) + L(a_R - a_L) - 2ts(R-L^2)]}{2t(R-L)^2}
$$

$$
= -\frac{(a_R - a_L)}{2t(R-L)^2}
$$

$$
\Rightarrow \lambda_{RL}a_L + (1-\lambda_{RL})a_R =
$$

$$
\frac{[2ta_R(R-L)(s-L) - a_R(a_R - a_L) + 2ta_L(R-L)(R-s) + a_L(a_R - a_L)]}{2t(R-L)^2}
$$

$$
= \frac{Ra_L - La_R + s(a_R - a_L)}{(R-L)} - \frac{(a_R - a_L)^2}{2t(R-L)^2}.
$$

Hence

$$
\begin{aligned}
U_{RL} &= \int_{s_P}^{s_R} \{\underline{v} - t[\lambda_{RL}L(1-\lambda_{RL})R - s]^2 - [\lambda_{RL}a_L + (1-\lambda_{RL})a_R]\} f(s)\mathrm{d}s \\
&= \int_{s_P}^{s_R} \left\{\underline{v} - t\left[\frac{a_R - a_L}{2t(R-L)}\right]^2 - \frac{s(a_R - a_L)}{(R-L)} + \frac{(a_R - a_L)^2}{2t(R-L)^2}\right\} f(s)\mathrm{d}s \\
&= \underline{v}(s_R - s_P) - \left[\frac{a_R - a_L}{(R-L)}\right]^2 \left(\frac{1}{4t} - \frac{1}{2t}\right)(s_R - s_P) - \frac{(a_R - a_L)}{2(R-L)}(s_R^2 - s_P^2) \qquad (A36)\\
&= \underline{v}(s_R - s_P) + t\left[\frac{a_R - a_L}{2t(R-L)}\right]^2 (s_R - s_P) - \frac{(a_R - a_L)}{2(R-L)}(s_R^2 - s_P^2) \\
&= \underline{v}(s_R - s_P) + t\left[\frac{a_R - a_L}{2t(R-L)}\right]^2 (R-L) \\
&\quad - \frac{(a_R - a_L)}{2(R-L)}\left[(R+L)(R-L) + \frac{(a_R - a_L)}{t}\right] \\
&= \underline{v}(s_R - s_P) + \frac{(a_R - a_L)^2}{4t(R-L)} - \frac{(a_R - a_L)^2}{2t(R-L)} - \frac{1}{2}(a_R - a_L)(R+L).
\end{aligned}
$$

From this,

$$\begin{aligned}
U_{RL} &= \int_{s_P}^{s_R} \{\underline{v} - t[\lambda_{RL} L(1-\lambda_{RL})R - s]^2 - [\lambda_{RL} a_L + (1-\lambda_{RL}) a_R]\} f(s)\mathrm{d}s \\
&= \int_{s_P}^{s_R} \left\{ \underline{v} - t\left[\frac{a_R - a_L}{2t(R-L)}\right]^2 - \frac{s(a_R - a_L)}{(R-L)} + \frac{(a_R - a_L)^2}{2t(R-L)^2} \right\} f(s)\mathrm{d}s \\
&= \underline{v}(s_R - s_P) - \left[\frac{a_R - a_L}{(R-L)}\right]^2 \left(\frac{1}{4t} - \frac{1}{2t}\right)(s_R - s_P) - \frac{(a_R - a_L)}{2(R-L)}(s_R^2 - s_P^2) \\
&= \underline{v}(s_R - s_P) + t\left[\frac{a_R - a_L}{2t(R-L)}\right]^2 (s_R - s_P) - \frac{(a_R - a_L)}{2(R-L)}(s_R^2 - s_P^2) \\
&= \underline{v}(s_R - s_P) + t\left[\frac{a_R - a_L}{2t(R-L)}\right]^2 (R-L) \\
&\quad - \frac{(a_R - a_L)}{2(R-L)}\left[(R+L)(R-L) + \frac{(a_R - a_L)}{t}\right] \\
&= \underline{v}(s_R - s_P) + \frac{(a_R - a_L)^2}{4t(R-L)} - \frac{(a_R - a_L)^2}{2t(R-L)} - \frac{1}{2}(a_R - a_L)(R+L).
\end{aligned} \tag{A37}$$

Hence,

$$\begin{aligned}
U_{RL} &= \underline{v}(s_R - s_P) - \frac{1}{2}(a_R - a_L)(R+L) - \frac{(a_R - a_L)^2}{4t(R-L)} \\
&= \underline{v}(s_R - s_P) - \frac{1}{2}(a_R + a_L)(s_R - s_P) - \frac{(a_R - a_L)^2}{4t(R-L)} \\
&\quad + \frac{1}{2}[(a_R + a_L)(R-L) - (a_R - a_L)(R+L)] \\
&= \underline{v}(s_R - s_P) - \frac{1}{2}(a_R + a_L)(s_R - s_P) - \frac{(a_R - a_L)^2}{4t(R-L)}.
\end{aligned} \tag{A38}$$

Finally, there are consumers listening only to R with similarly derived surplus of:

$$\begin{aligned}
U_R &= \int_{s_R}^{1} [\underline{v} - t(s-R)^2 - a_R] f(s)\mathrm{d}s \\
&= \underline{v}(1 - s_R) - a_R(1 - s_R) - \frac{t}{3}[(1-R)^3 - (s_R - R)^3].
\end{aligned} \tag{A39}$$

Putting these terms all together we get overall consumer welfare of:

$$\begin{aligned}
U &= U_P + U_{PL} + U_{LR} + U_R \\
&= \underline{v} - \frac{1}{2}[a_L(s_p - s_0) + (a_L + a_R)(s_R - s_p) + 2a_R(1 - s_R)] \\
&\quad - \frac{1}{3}t[s_0^3 + (1-R)^3 - (s_R - R)^3] - \frac{a_L^2}{4tL} - \frac{(a_R - a_L)^2}{4t(R-L)} \\
&= \underline{v} - A - \frac{1}{3}t[s_0^3 + (1-R)^3 - (s_R - R)^3] - \frac{a_L^2}{4tL} - \frac{(a_R - a_L)^2}{4t(R-L)}
\end{aligned} \tag{A40}$$

where A again denotes total advertising heard:

$$A = \frac{1}{2}[a_L(s_p - s_0) + (a_R + a_L)(s_R - s_p)] + a_R(1 - s_R).$$

Summing the profits of radio stations and the surplus of advertisers gives:

$$S_L + S_R + \pi_L + \pi_R + \pi_P = A - 3F.$$

Thus, with a non-commercial public radio station playing only local music we get welfare of:

$$\begin{aligned} v^R &= U + \pi_P + \pi_L + \pi_R + S_L + S_R + \rho M_L \\ &= \underline{v} + \rho M_L - \frac{1}{3}t\left\{\left(\frac{a_L}{2tL}\right)^3 + (1-R)^3 - \left[\frac{(a_R - a_L)}{2t(R-L)}\right]^3\right\} - \frac{a_L^2}{4tL} - \frac{(a_R - a_L)^2}{4t(R-L)} - 3F \\ &\cong \underline{v} + 0.5830\rho - 0.0071t - 3F. \end{aligned} \tag{22}$$

If the regulator's objective is to increase the amount of local music heard, we have seen that this policy raises it from 50% to around 58%. If we chose the local content requirement that yielded the same amount of local music, setting the expression for M_L in (18) equal to a little over 58%, we would need a requirement of around 28% which would yield welfare of $\underline{v} + 0.5830\rho - 0.0098t - 2F$. The comparison of the two options then depends on the fixed costs of establishing the public station.

Calculations for a Public Station and a Single Commercial Station

Suppose a public non-commercial station is free to choose its location, to maximise welfare, in competition with a single commercial station that seeks to maximise its profits. Denote the public station (and its location) by P and assume that $P < R$. As $a_P = 0$ so (6) yields:

$$\begin{aligned} x_P(P, R, a_R) &= \frac{1}{2}\left[\frac{a_R}{t(R-P)} + (R+P)\right] \\ x_R(P, R, a_R) &= \frac{1}{2}\left[2 - (R+P) - \frac{a_R}{t(R-P)}\right]. \end{aligned} \tag{A41}$$

From (7), the demand for advertising at R then solves:

$$p_R = x_R + a_R\frac{\mathrm{d}x_R}{\mathrm{d}a_R} = \frac{1}{2}\left[2 - (R+P) - \frac{2a_R}{t(R-P)}\right]$$

(from (A41)) which inverts for:

$$a_R = \frac{1}{2}t(R-P)[2 - (R+P) - 2p_R].$$

Choosing p_R to maximise $\pi_R = a_R p_R$ then gives the following:

$$p_R = \frac{1}{4}[2 - (R+P)] \Rightarrow a_R = \frac{1}{4}t(R-P)[2 - (R+P)].$$

Turning to the locational decisions of the two stations, station R chooses R to maximise the following expression for profit:

$$\underset{\{R\}}{\text{Max}}\, \pi_R = a_R p_R = \frac{1}{16} t(R-P)[2-(R+P)]^2 - F$$
$$\Rightarrow R = \frac{1}{3}(2+P).$$

The public firm, however, chooses P to maximise welfare:

$$\underset{\{P\}}{\text{Max}}\, W = \underline{v} - 2F - \frac{t}{3}[p^3 + (1-R)^3] - \frac{4t}{81}(R-P)[1-(R+P)]$$
$$\Rightarrow -69P + 8(R-2)P + 4(1-R^2) = 0$$
$$\Rightarrow R(P).$$

Solving R and P jointly from these two first-order conditions yields the following values for locations and these can be substituted into the relevant expressions for market shares, local music heard ($M_L = x_P(1-P) + x_R(1-R)$), welfare and advertising to yield:

$$x_P \cong 0.5558 \quad M_L \cong 0.6253$$
$$x_R \cong 0.4442 \quad a_R \cong 0.1754 \quad p_R \cong 0.2961.$$

By setting our earlier expression for local music heard under a quota – $M_{L\gamma} = \frac{1}{18}(5\gamma^2 - 14\gamma + 18)$ – equal to the level heard in this context ($\cong$0.6253) we can solve for the local content-equivalent quota: $\gamma \cong 0.1732$. Evaluating welfare under the quota with such a value of γ gives $W_\gamma \cong \underline{v} - 2F - 0.1943t$.