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WHY DOES THE YIELD CURVE PREDICT OUTPUT AND INFLATION?

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Appendix: Detailed Expressions and Derivations

A.1. *Elements of the Transition Matrix*

Exact expressions for the elements of the 3 × 3 transition matrix **T** of Section 1 are given below.

$$T = \begin{bmatrix} 1 & a & 0 \\ \dfrac{(2-g_\pi)b_2}{2+(g_y-a)b_2} & \dfrac{(2-g_\pi)b_2 a + 2b_1}{2+(g_y-a)b_2} & \dfrac{-(1+g_r)b_2}{2+(g_y-a)b_2} \\ \dfrac{g_\pi(2-ab_2)+2g_y b_2}{2+(g_y-a)b_2} & \dfrac{ag_\pi(2-ab_2)+2g_y(ab_2+b_1)}{2+(g_y-a)b_2} & \dfrac{g_r(2-ab_2)-g_y b_2}{2+(g_y-a)b_2} \end{bmatrix}.$$

A.2. *Bounds for Inflation and Output Reaction Parameters*

Bounds for g_π and g_y required for the system to be stationary, $B_\pi(g_r, g_y)$ and $B_y(g_r)$, have the following forms.

$$B_\pi = 2b_1(2-2b_1-ab_2)g_r^2 + (2+2b_1+ab_2)(2b_1+ab_2-2-b_2 g_y)g_r$$
$$+ (2-ab_2+b_2 g_y)(2-2b_1-ab_2+2b_2 g_y)/ab_2(2-ab_2+2b_1 g_r+b_2 g_y)$$

$$B_y = a - \frac{(1-g_r)(1-b_1)}{b_2}.$$

A.3. *Derivation of Optimal Policy Rule*

This Section sketches the solution to the monetary authority's general optimisation problem (accommodating either strict or flexible inflation targeting) in the backward-looking case. To simplify the solution procedure, we first recognise the following features of the problem. First, the optimal values of the policy parameters (g_r, g_π and g_y) in the general solution to the problem with uncertainty are the same as under certainty (certainty equivalence), so we assume the deterministic case.[1] Second, the optimal policy parameters are independent of the inflation target, which we set to zero. Third, the optimal value of the loss function at time t is a quadratic function of the endogenous variables. Fourth, there are no adjustment costs, so adjustment to the desired level of the interest rate is instantaneous, hence $g_r = 0$. Finally note that the optimal feedback rule (r_t as a function of the endogenous variables) for this problem is linear, so the reaction function given in (5) has the optimal linear form.

In the model, a change in the policy instrument r_t affects output in the next period, which in turn affects inflation with a further one-period lag. This structure allows for sequential

[1] See Chow (1975, Chapters 7 and 8) for a complete discussion of the solution to linear-quadratic optimal control problems, including the features listed here.

optimisation, first setting output to control inflation, and then the interest rate to control both output and inflation.

To find the solution with respect to output, express the optimisation problem in terms of the following Bellman equation,

$$L_t = \min_{y_t}\left\{\frac{1}{2}\left[(1-w)\pi_t^2 + wy_t^2\right] + \delta L_{t+1}\right\}, \tag{36}$$

where L_t is the minimum loss function. As noted above, the optimal L_t is quadratic in the endogenous variable, which in this case is π_t.

Let

$$L_t = \frac{1}{2}q\pi_t^2. \tag{37}$$

The first order condition is then

$$wy_t + \delta qa\pi_{t+1} = 0. \tag{38}$$

Since q is unknown, we use the envelope theorem to construct a second relationship from which to calculate q. Specifically $\partial L_t/\partial\pi_t = \partial\{\}/\partial\pi_t|y_t^*$, where the brackets correspond to the bracketed expression in (36) and $|y_t^*$ indicates evaluation at the optimal value of y_t. Thus,

$$q\pi_t = (1-w)\pi_t + \delta q\pi_{t+1}. \tag{39}$$

The Phillips curve relationship $\pi_{t+1} = \pi_t + ay_t$ from (2) may be substituted in expressions (38) and (39), respectively, to obtain

$$y_t^* = \frac{-\delta aq}{w+\delta a^2 q}\pi_t = \frac{(1-\delta)q-(1-w)}{\delta aq}\pi_t \equiv m\pi_t. \tag{40}$$

The coefficients of π_t must be equal and this equality may be used to solve for q and m. Expression (39) may also be solved directly for π_{t+1} to obtain

$$\pi_{t+1} = \frac{w}{w+\delta a^2 q}\pi_t \equiv \theta\pi_t. \tag{41}$$

After solving for q using (40), we have

$$\theta = 2w/(1+\delta-\delta a^2)w + \delta a^2 + \sqrt{(1-\delta-\delta a^2+2\delta a)(1-\delta-\delta a^2-2\delta a)w^2 + (1+\delta-\delta a^2)2\delta a^2 w + \delta^2 a^4}. \tag{42}$$

and $m = (\theta-1)/a$. Inspection of (42) shows that $0 \le \theta(w) \le 1$, $\theta'(w) > 0$, $\theta(0) = 0$, and $\theta(1) = 1$, as noted in the text.

We now equate two expressions for y_{t+1}, namely the optimal rule from (40) and the IS equation from (1):

$$m\pi_{t+1} = b_1 y_t - \frac{1}{2}b_2(r_t + r_{t+1} - \pi_{t+1} - \pi_{t+2}). \tag{43}$$

Use the solution of the five-equation system (11)–(15) to substitute for π_{t+1}, π_{t+2} and r_{t+1} in terms of π_t, y_t and r_t, and solve the resulting expression for r_t, obtaining an equation of the form $r_t = c_\pi\pi_t + c_y y_t$. At the optimum, $c_\pi = g_\pi$ and $c_y = g_y$, and these two equations may be used to obtain the optimal values of the two parameters,

$$g_\pi = 1 + \frac{2(1+b_1)}{ab_2}(1-\mu), g_y = a + \frac{2(1+2b_1)}{b_2}(1-\mu) + \frac{2b_1}{b_2}\mu, \tag{44}$$

where $\mu \equiv 2\theta/(1+\theta)$.