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Technical Appendix to

MEMORY AND ANTICIPATION

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Appendix

Proof of Theorem 2. We start with some preliminaries. First, define

$$\lambda(p,\beta) = \frac{(1-p)\beta}{1-p\beta}$$

and

$$\xi(p,\beta) = (1+\delta)[pa + (1-p)b] - [pc + (1-p)d][\delta + \lambda(p,\beta)] - [1-\lambda(p,\beta)]d.$$

Note that

$$\xi(0,\beta) = (1+\delta)b - d(\delta+\beta) - d(1-\beta) = (1+\delta)(b-d) < 0$$

and

$$\xi(1,\beta) = (1+\delta)a - c\delta - d = (a-d) - \delta(c-a) > 0$$

(where the last inequality follows from our assumption that $\delta < (a-d)/(c-a)$). Consequently, we know that, for all $\beta \in (0, 1)$, there exists $p^* \in (0, 1)$ satisfying $\xi(p^*, \beta) = 0$. We will write this as $p^*(\beta)$.

We consider equilibrium strategies of the following form: for some $\beta \in (0, 1)$,

$$\begin{array}{l} \text{If } \omega \le \beta, \text{ play } C \text{ with probability } p^*(\beta), \text{ and } N \text{ with probability } 1-p^*(\beta) \\ \text{If } \omega > \beta, \text{ play } N \end{array} \tag{19}$$

Step 1. For all $\beta \in (0, 1)$, the strategies described in (19) constitute an equilibrium.

First suppose that $\omega \le \beta$ is observed. We compute the payoff from each choice and verify that i is indifferent between C and N, which means he is willing to randomise.

If i chooses C, his payoff is

$$E(u_i \mid C) + \delta E(u_i \mid C, \omega).$$

Let us start with $E(u_i \mid C, \omega)$. Knowing $\omega \le \beta$, i can infer from j's equilibrium strategy that j has chosen C with probability $p^*(\beta)$, so

$$E(u_i \mid C, \omega) = p^*(\beta)a + [1-p^*(\beta)]b.$$

Now consider $E(u_i \mid C)$. Recalling that he has chosen C, i can infer from his own equilibrium strategy that $\omega \le \beta$, in which case he concludes that j has chosen C with probability $p^*(\beta)$, so

$$E(u_i \mid C) = p^*(\beta)a + [1-p^*(\beta)]b.$$

Thus,

$$\mathrm{E}(u_i \mid C) + \delta\mathrm{E}(u_i \mid C, \omega) = (1+\delta)\{p^*(\beta)a + [1-p^*(\beta)]b\}. \tag{20}$$

If i chooses N, his payoff is

$$\mathrm{E}(u_i \mid N) + \delta\mathrm{E}(u_i \mid N, \omega).$$

Let us start with $\mathrm{E}(u_i \mid N, \omega)$. Knowing $\omega \leq \beta$, i can infer from j's equilibrium strategy that j has chosen C with probability $p^*(\beta)$, so

$$\mathrm{E}(u_i \mid N, \omega) = p^*(\beta)c + [1-p^*(\beta)]d.$$

Now consider $\mathrm{E}(u_i \mid N)$. In stage 4, i will recall that he has chosen N. Given his equilibrium strategy, the unconditional probability of $\omega \leq \beta$ and choosing N is $\beta[1-p^*(\beta)]$, and the unconditional probability of $\omega > \beta$ and choosing N is $1-\beta$. Therefore, the probability of $\omega \leq \beta$ conditional upon observing N is

$$\frac{[1-p^*(\beta)]\beta}{[1-p^*(\beta)]\beta + (1-\beta)} = \frac{[1-p^*(\beta)]\beta}{1-p^*(\beta)\beta} = \lambda[p^*(\beta), \beta].$$

Player i can infer from recalling N and his own equilibrium strategy that his expected payoff is

$$\mathrm{E}(u_i \mid N) = \lambda[p^*(\beta), \beta]\{p^*(\beta)c + [1-p^*(\beta)]d\} + \{1-\lambda[p^*(\beta), \beta]\}d.$$

Thus,

$$\begin{aligned}&\mathrm{E}(u_i \mid N) + \delta\mathrm{E}(u_i \mid N, \omega)\\ &= \{p^*(\beta)c + [1-p^*(\beta)]d\}\{\delta + \lambda[p^*(\beta), \beta]\} + \{1-\lambda[p^*(\beta), \beta]\}d.\end{aligned} \tag{21}$$

Using (20) and (21), we see that

$$\mathrm{E}(u_i \mid C) + \delta\mathrm{E}(u_i \mid C, \omega) - \mathrm{E}(u_i \mid N) - \delta\mathrm{E}(u_i \mid N, \omega) = \xi[p^*(\beta), \beta]. \tag{22}$$

But $\xi[p^*(\beta), \beta] = 0$ by construction. This means that, when $\omega \leq \beta$, i is indifferent between C and N, as required.

Second suppose that $\omega > \beta$ is observed. We need to verify that i weakly prefers to pick N.

If i chooses C, his payoff is

$$\mathrm{E}(u_i \mid C) + \delta\mathrm{E}(u_i \mid C, \omega).$$

Clearly, exactly as above, $\mathrm{E}(u_i \mid C) = p^*(\beta)a + [1-p^*(\beta)]b$. What about $\mathrm{E}(u_i \mid C, \omega)$? Since $\omega > \beta$, i can infer from j's equilibrium strategy that j will play N. Therefore $\mathrm{E}(u_i \mid C, \omega) = b$, and

$$\mathrm{E}(u_i \mid C) + \delta\mathrm{E}(u_i \mid C, \omega) = p^*(\beta)a + [1-p^*(\beta)]b + \delta b. \tag{23}$$

If i chooses N, his payoff is

$$\mathrm{E}(u_i \mid N) + \delta\mathrm{E}(u_i \mid N, \omega).$$

Clearly, exactly as above,

$$\mathrm{E}(u_i \mid N) = \lambda[p^*(\beta), \beta]\{p^*(\beta)c + [1-p^*(\beta)]d\} + \{1-\lambda[p^*(\beta), \beta)]\}d.$$

What about $\mathrm{E}(u_i \mid N, \omega)$? Since $\omega > \beta$, i can infer from j's equilibrium strategy that j will play N. Therefore $\mathrm{E}(u_i \mid N, \omega) = d$, and

$$\begin{aligned}\mathrm{E}(u_i \mid N) + \delta\mathrm{E}(u_i \mid N, \omega) = {}&\lambda[p^*(\beta), \beta]\{p^*(\beta)c + [1-p^*(\beta)]d\}\\ &+ \{1-\lambda[p^*(\beta), \beta]\}d + \delta d.\end{aligned} \tag{24}$$

Combining (23) and (24), we see that

$$\begin{aligned}&\mathrm{E}(u_i \mid N) + \delta\mathrm{E}(u_i \mid N, \omega) - \mathrm{E}(u_i \mid C) - \delta\mathrm{E}(u_i \mid C, \omega)\\ &= -\xi[p^*(\beta), \beta] - \delta p^*(\beta)(c-d) + \delta p^*(\beta)(a-b)\\ &= \delta p^*(\beta)[(d-b)-(c-a)] > 0\end{aligned}$$

where the last inequality follows from Assumption P2. But this means that i, upon observing $\omega > \beta$, strictly prefers to choose N, as specified by his strategy.

Step 2: $\lim_{\beta\uparrow 1} p^*(\beta) = 1$.

Suppose not. Then we can find a sequence $\beta_k \uparrow 1$ as $k \to \infty$ such that $\lim_{k\to\infty} p^*(\beta_k) = p_0 < 1$. We know that $\lim_{k\to\infty} \xi[p^*(\beta_k), \beta_k] = 0$ (since it is zero for each k).

By inspection, $\xi(p, \beta)$ is continuous in some neighbourhood of $(p, \beta) = (p_0, 1)$ (since $p_0 < 1$). Therefore,

$$\begin{aligned}\lim_{k\to\infty} \xi[p^*(\beta_k), \beta_k] &= \xi[\lim_{k\to\infty} p^*(\beta_k), \lim_{k\to\infty} \beta_k]\\ &= \xi(p_0, 1)\\ &= (1+\delta)[p_0 a + (1-p_0)b] - (1+\delta)[p_0 c + (1-p_0)d]\\ &= (1+\delta)[p_0(a-c) + (1-p_0)(b-d)] < 0,\end{aligned}$$

which is a contradiction.

Combining steps 1 and 2 gives us the desired conclusion.