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Technical Appendix to

INTERTEMPORAL PRICE CAP REGULATION UNDER UNCERTAINTY

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Appendix

This Appendix gives an outline of the analysis involved in obtaining results 2–4. For an exposition of smooth pasting/boundary conditions, see Dixit (1993) and Dumas (1991). Detailed step by step derivations are given in the appendices to the original working papers (Dobbs, 2000; 2001). To reduce notational clutter, time subscripts and function arguments are dropped in what follows (where this results in no loss of intelligibility).

A1. *Competition/Monopoly without a Price Cap*

The arbitrage equation (Dixit, 1993, p. 15), from (10) is that

$$(r+\theta)v\mathrm{d}t = p\mathrm{d}t + \mathrm{E}(\mathrm{d}v). \tag{A.1}$$

Applying Itô's lemma and simplifying, this yields the following fundamental differential equation:

$$\tfrac{1}{2}\eta^2\sigma^2x^2\psi'' + \left(\mu_p+\delta\right)x\psi' - (\delta+\theta+r)\psi + x = 0. \tag{A.2}$$

The general solution to (A.2) can be shown to have the form

$$\psi(x) = B_0x + B_1x^{\lambda_1} + B_2x^{\lambda_2}, \tag{A.3}$$

where B_0, λ_1, λ_2, R_1, R_2 are given by (12)–(16). The arbitrary constants B_1, B_2 are determined by boundary conditions. Given $\lambda_2 < 0$ from (14), as $x \to 0$, if value is to be finite, it must be that $B_2 = 0$. The other constant B_1 is determined by an analysis of smooth pasting conditions at the boundary (at which new investment is triggered). This is now done in turn for the competitive and monopoly cases.

A.1.1. *Competitive industry under uncertainty*

In a competitive industry, new investment occurs when expected value for a unit of capacity rises to equal the unit cost of investment. This value matching condition occurs at a time $\tilde{t}$ at which the price $p_{\tilde{t}}$ reaches the level $p_{\tilde{t}} = \xi_u K_{\tilde{t}}$ (equivalently $x_{\tilde{t}} = \xi_u$) where ξ_u is the competitive uncertainty relative price that triggers new investment:

$$v(p_{\tilde{t}}, K_{\tilde{t}}) = \psi(p_{\tilde{t}}/K_{\tilde{t}})K_{\tilde{t}} = K_{\tilde{t}} \Rightarrow \psi(p_{\tilde{t}}/K_{\tilde{t}}) = 1 \quad \Rightarrow \psi(\xi_u) = 1, \tag{A.4}$$

Smooth pasting additionally requires the following 'first order condition' to hold;

$$\frac{\mathrm{d}}{\mathrm{d}p}[v(p,K) - K] = \frac{\mathrm{d}}{\mathrm{d}p}\Big[\psi(p/K)K\Big] = \psi'(p/K) = 0 \Rightarrow \psi'(\xi_u) = 0. \tag{A.5}$$

The conditions (A.4) and (A.5) can be used to determine B_1 and ξ_u. After some manipulation the result for ξ_u is that

$$\xi_u = \left(\frac{\lambda_1}{\lambda_1 - 1}\right)\left(\theta + r - \mu_p\right), \tag{A.6}$$

which is the first part of Result 2(*i*) in the paper. After further routine algebra, it is possible to show that (A.6) can also be re-expressed as

$$\xi_u = \left(\frac{1-\lambda_2}{\lambda_2}\right)(\theta + r + \delta) = \left(\frac{1-\lambda_2}{\lambda_2}\right)\xi_c \tag{A.7}$$

as reported in Result 2(*i*).

A.1.2. *Monopoly under uncertainty*

In the monopoly case, investment commences at a time $\tilde{t}$ at which price $p_{\tilde{t}}$ reaches the level $p_{\tilde{t}} = \xi_M K_{\tilde{t}}$, where ξ_M is the relative price at which new capacity is added. Since ξ_M is a free choice by the firm, smooth pasting involves first and second derivative conditions; see Dumas (1991). The first derivative condition is that, with respect to the control variable, the rate of change of value should just equal the rate of change of cost;

$$\partial V(p_{\tilde{t}}, K_{\tilde{t}}, Q_{\tilde{t}})/\partial p_{\tilde{t}} = \partial(K_{\tilde{t}} Q_{\tilde{t}})/\partial p_{\tilde{t}} \tag{A.8}$$

where $Q_{\tilde{t}} = A_{\tilde{t}} p_{\tilde{t}}^{\gamma}$. The second derivative condition is

$$\partial^2 V(p_{\tilde{t}}, K_{\tilde{t}}, Q_{\tilde{t}})/\partial p_{\tilde{t}}^2 = \partial^2(K_{\tilde{t}} Q_{\tilde{t}})/\partial p_{\tilde{t}}^2. \tag{A.9}$$

These conditions imply:

$$\gamma[\psi(\xi_M) - 1] + \xi_M \psi'(\xi_M) = 0, \tag{A.10}$$

$$\gamma(\gamma-1)[\psi(\xi_M) - 1] + (\gamma - 1)\xi_M \psi\prime(\xi_M) + [(1+\gamma)\xi_M \psi\prime(\xi_M) + \xi_M^2 \psi\prime\prime(\xi_M)] = 0. \tag{A.11}$$

These serve to define the unknowns B_1 and ξ_M. After some routine algebra, the solution for ξ_M can be simplified to give

$$\xi_M = \left(\frac{\gamma}{1+\gamma}\right)\left(\frac{\lambda_2 - 1}{\lambda_2}\right)(\theta + r + \delta), \tag{A.12}$$

as in Result 2(*ii*).

A2. *Price Capped Monopoly*

Let ψ denote the solution when there is zero investment and no price constraint, whilst ψ_2 denotes the solution when the price constraint applies but there is no investment. There are now three regimes; unconstrained with no investment, price constrained with no investment and price constrained with investment. The solution in the first two regimes is first discussed, followed by an analysis of the smooth pasting conditions at the interfaces between the regimes.

A.2.1. *Regime 1: Unconstrained price, no investment*

The solution here is naturally identical to that already established for the unconstrained monopoly firm – that is, the solution is given by (11) where B_0 is given by (12). As before,

note that, as $x \to 0$, if $v(p, K)$ is to be finite, it must be that $B_2 = 0$. The constant B_1 in this new problem is determined via an analysis of boundary conditions (see below).

A.2.2. *Regime 2: Price constrained, no investment*
In this region, the price cap binds and $p_t = \bar{\xi}K_t$; the arbitrage equation becomes

$$(r+\theta)v\mathrm{d}t = \bar{\xi}K\mathrm{d}t + \mathrm{E}(\mathrm{d}v). \tag{A.13}$$

The analysis parallels that for (A.2); it yields (compare with (A.2)):

$$\tfrac{1}{2}\eta^2\sigma^2x^2\psi'' + \left(\mu_p + \delta\right)x\psi' - (\delta+\theta+r)\psi + \bar{\xi} = 0, \tag{A.14}$$

as the fundamental differential equation. The solution in regime 2 is denoted ψ_2 and this is given as

$$\psi_2(x) = \left[\bar{\xi}/(\delta+\theta+r)\right] + C_1x^{\lambda_1} + C_2x^{\lambda_2}, \tag{A.15}$$

where the arbitrary constants C_1, C_2 are determined by a consideration of boundary conditions.

A.2.3. *Analysis of transition boundary conditions:*
Let $\tilde{t}_1$ denote a hitting time at which there is a transition between the regimes 1 and 2 whilst $\tilde{t}_2$ denotes a hitting time at which there is a transition between the regimes 2 and 3 (at which new investment commences).

A.2.4. *Regime 1/2 boundary:*
At this boundary, by definition, the price cap binds, so $x_{\tilde{t}_1} = \bar{\xi}$. As far as the firm is concerned, $\bar{\xi}$ is exogenous; as a consequence, smooth pasting involves matching value and first derivatives for the solutions as they meet at the boundary (Dumas, 1991). Since $v(x_t) = \psi(x_t)K_t$, this requires

$$\psi(\bar{\xi}) = \psi_2(\bar{\xi}), \tag{A.16}$$

$$\psi'(\bar{\xi}) = \psi_2'(\bar{\xi}), \tag{A.17}$$

where, from the definitions of ψ and ψ_2, these are calculated as $\psi(\bar{\xi}) = B_0\bar{\xi} + B_1\bar{\xi}^{\lambda_1}$, $\psi'(\bar{\xi}) = B_0 + \lambda_1B_1\bar{\xi}^{\lambda_1-1}$, $\psi_2(\bar{\xi}) = \left[\bar{\xi}/(\theta+r+\delta)\right] + C_1\bar{\xi}^{\lambda_1} + C_2\bar{\xi}^{\lambda_2}$ and $\psi_2'(\bar{\xi}) = \lambda_1C_1\bar{\xi}^{\lambda_1-1} + \lambda_2C_2\bar{\xi}^{\lambda_2-1}$.

A.2.5. *Regime 2/3 boundary:*
Here ξ denotes the relative trigger market clearing price at which the firm would choose to start to invest when the firm is subject to a price cap. Since the choice of ξ is free, the smooth pasting conditions at $\tilde{t}_2$ require the first and second derivatives of the value function in regime 2 to satisfy equivalent conditions to those specified above in the unconstrained monopoly case. That is,

$$\gamma\left[\psi_2(\xi) - 1\right] + \xi\psi_2'(\xi) = 0 \tag{A.18}$$

$$\gamma(\gamma-1)\left[\psi_2(\xi) - 1\right] + (\gamma-1)\xi\psi_2'(\xi) + \left[(1+\gamma)\xi\psi_2'(\xi) + \xi^2\psi_2''(\xi)\right] = 0 \tag{A.19}$$

where the derivatives are calculated as in the analysis at the regime 1/2 boundary (but evaluated at ξ).

A.2.6. *Analysis of smooth pasting conditions:*
After some routine algebra, it is possible to solve (A.16)–(A.19) to determine the arbitrary constants B_1, C_1, C_2 and the value of ξ (as a function of $\bar{\xi}$ and the other parameters in the

problem). The solution for ξ is given in Result 3 (the full 'step by step' derivation being given in the Appendix to the working paper (Dobbs, 2001).

A3. *Proof for Result 4*

The formula for $\xi(\bar{\xi})$ in Result 3 was

$$\xi = \left[(\bar{\xi} - \xi_c)\xi_M \bar{\xi}^{\lambda_2 - 1}/(\xi_M - \xi_c)\right]^{1/\lambda_2}. \tag{A.20}$$

Differentiating with respect to $\bar{\xi}$ gives

$$\frac{\mathrm{d}\xi}{\mathrm{d}\bar{\xi}} = (1/\lambda_2)\left[\left(\frac{\bar{\xi} - \xi_c}{\xi_M - \xi_c}\right)\xi_M \bar{\xi}^{\lambda_2 - 1}\right]^{(1/\lambda_2)-1}\left(\frac{\xi_M}{\xi_M - \xi_c}\right)\frac{\mathrm{d}}{\mathrm{d}\bar{\xi}}\left[(\bar{\xi} - \xi_c)\bar{\xi}^{\lambda_2 - 1}\right] \tag{A.21}$$

where, using the definition for ξ_u,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\bar{\xi}}\left[(\bar{\xi} - \xi_c)\bar{\xi}^{\lambda_2 - 1}\right] &= \frac{\mathrm{d}}{\mathrm{d}\bar{\xi}}\left(\bar{\xi}^{\lambda_2} - \xi_c \bar{\xi}^{\lambda_2 - 1}\right) = \left[\lambda_2 \bar{\xi}^{\lambda_2 - 1} - (\lambda_2 - 1)\xi_c \bar{\xi}^{\lambda_2 - 2}\right] \\ &= \lambda_2 \bar{\xi}^{\lambda_2 - 2}(\bar{\xi} - \xi_u)\end{aligned} \tag{A.22}$$

so

$$\frac{\mathrm{d}\xi}{\mathrm{d}\bar{\xi}} = \underbrace{\left[\left(\frac{\bar{\xi} - \xi_c}{\xi_M - \xi_c}\right)\xi_M \bar{\xi}^{\lambda_2 - 1}\right]^{(1/\lambda_2)-1}}_{(+)}\underbrace{\left(\frac{\xi_M}{\xi_M - \xi_c}\right)}_{(+)}\underbrace{\xi^{\lambda_2 - 2}}_{(+)}(\bar{\xi} - \xi_u) \tag{A.23}$$

hence

$$\frac{\mathrm{d}\xi}{\mathrm{d}\bar{\xi}} \gtreqless 0 \text{ as } \bar{\xi} \gtreqless \xi_u. \tag{A.24}$$

This completes the proof for Result 4(i). As $\bar{\xi} \downarrow \xi_c$, the term in brackets [] $\rightarrow$ 0 in (A.20); since $1/\lambda_2 < 0$, it follows that $\xi \rightarrow +\infty$, which is Result 4(ii). Letting $\bar{\xi} \rightarrow \xi_M$ in (A.20), clearly $\xi(\bar{\xi}) \rightarrow \xi_M$, which is Result 4($iii$). Setting $\bar{\xi} = \xi_u$, from (A.20),

$$\xi^{\lambda_2} = \left[(\xi_u - \xi_c)\xi_M \xi_u^{\lambda_2 - 1}/(\xi_M - \xi_c)\right]. \tag{A.25}$$

Now, $\xi(\xi_u) \gtreqless \xi_u$ as $\xi(\xi_u)^{\lambda_2} \lesseqgtr \xi_{+}^{\lambda_{\preccurlyeq}}$ (since $\lambda_2 < 0$). Using (A.25) this implies $\xi(\xi_u) \gtreqless \xi_u$ as

$$\begin{aligned}&(\xi_u - \xi_c)\xi_M \xi_u^{\lambda_2 - 1}/(\xi_M - \xi_c) \lesseqgtr \xi_{+}^{\lambda_{\preccurlyeq}} \\ &\Rightarrow (\xi_u - \xi_c)\xi_M \lesseqgtr \xi_{+}(\xi_{\triangle} - \xi_{\Supset}) \Rightarrow \xi_{\triangle} - \xi_{+} \gtreqless \lessdot.\end{aligned}$$

In fact $\xi_M - \xi_u > 0$ and hence $\xi(\xi_u) > \xi_u$, which is Result 4 (iv).

To establish Result 4 (v), first substitute in (A.25) using result 2, for $\xi_u = (\lambda_2 - 1)\xi_c/\lambda_2$ and $\xi_M = (\lambda_2 - 1)\xi_c/\lambda_2(1 + \eta)$ to get $\xi(\xi_u) = (1 + \eta\lambda_2)^{-1/\lambda_2}\xi_u$. From the definition for λ_2, note that $\mathrm{Lim}_{\sigma \rightarrow 0}\, \lambda_2 = -\infty$, and so $\mathrm{Lim}_{\sigma \rightarrow 0}(\lambda_2 - 1)/\lambda_2 = 1$. Hence from Result 3, $\mathrm{Lim}_{\sigma \rightarrow 0}\ \xi_u = \xi_c$. Also $\mathrm{Lim}_{\sigma \rightarrow 0}(1+\eta\lambda_2)^{-1/\lambda_2} = \mathrm{Lim}_{\lambda_2 \rightarrow -\infty}(1+\eta\lambda_2)^{-1/\lambda_2} = 1$. Hence

$$\mathrm{Lim}_{\sigma \rightarrow 0}\xi(\xi_u) = \mathrm{Lim}_{\sigma \rightarrow 0}(1 + \eta\lambda_2)^{-1/\lambda_2}\mathrm{Lim}_{\sigma \rightarrow 0}\xi_u = \xi_c \tag{A.26}$$

which is Result 4 (v).