The Economic Journal, **114** (*April*), © Royal Economic Society 2004. Published by Blackwell Publishing, 9600 Garsington Road, Oxford OX4 2DQ, UK and 350 Main Street, Malden, MA 02148, USA.

Technical Appendix to

INNOVATION BY LEADERS*

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ECONOMIC JOURNAL, vol. 114 (*April*), pp. 281–303

Appendix A. Proofs

Proof of Proposition 1. Symmetry between the entrants in the second stage implies the equilibrium system:

$$f(\cdot) \equiv [h'(z)V - 1][r + nh(z) + h(z^L)] - h'(z)[h(z)V - z] = 0$$

$$g(\cdot) \equiv \left[h'(z^L)V - 1\right][r + nh(z) + h(z^L)] - \left[h'(z^L) + \frac{\partial nh(z)}{\partial z^L}\right]\left[h(z^L)V + \pi - z^L\right] = 0$$

with

$$\frac{\partial nh(z)}{\partial z^L} = \frac{\partial n}{\partial z^L} h(z) + nh'(z)\phi'(z^L)$$

where

$$\phi'(z^L) = \frac{-[h'(z^L)V - 1]h'(z^L)}{h'(z)\{V[r + (n-1)h(z) + h(z^L)] + z\}}.$$

Since $\partial\phi'(z^L)/\partial r < 0$, $\partial\phi'(z^L)/\partial\pi = 0$ and $\partial\phi'(z^L)/\partial n > 0$, while $\partial\phi'(z^L)/\partial V$ is ambiguous, by totally differentiating the system above we obtain the comparative statics for $x = r$, π, n, V:

$$\begin{bmatrix} \frac{dz}{dx} \\ \\ \frac{dz^L}{dx} \end{bmatrix} = \frac{-1}{\Delta}\begin{bmatrix} g_{z^L} & -f_{z^L} \\ -g_z & f_z \end{bmatrix}\begin{bmatrix} f_x \\ g_x \end{bmatrix}$$

where $\Delta \equiv f_z g_{z^L} - f_{z^L} g_z > 0$ by assumption of stability, and

$$f_z = h''(z)\left\{V[r + (n-1)h(z) + h(z^L)] + z\right\} + (n-1)h'(z)[h'(z)V - 1] < 0$$

$$f_{z^L} = h'(z^L)[h'(z)V - 1] > 0$$

* I am extremely grateful to Robert Barro for his support and comments at critical stages and to the Editor, David de Meza, for many important suggestions. I benefited from interesting comments by Daron Acemoglu, Philippe Aghion, Karina Firme, David Laibson, Greg Mankiw, Indira Pottebaum, Jennifer Reinganum, Joseph Zeira, two anonymous referees and seminar participants at M.I.T. and Harvard University. This paper was written at the National Bureau of Economic Research while I was a PhD student at Harvard University. I am grateful to NBER for providing a very exciting research environment.

$$f_r = h'(z)V - 1 > 0$$

$$f_\pi = 0$$

$$f_n = h'(z)[h'(z)V - 1] > 0$$

$$f_V = h'(z)[r + (n-1)h(z) + h(z^L)] > 0$$

$$g_z = nh'(z^L)[h'(z^L)V - 1] - nh''(z)\phi'(z^L)\left[h(z^L)V + \pi - z^L\right] > 0$$

$$g_{z^L} = h''(z^L)V[r + nh(z) - \pi + z^L] - nh'(z)\phi''(z^L)\left[h(z^L)V + \pi - z^L\right] < 0$$

$$g_r = h'(z^L)V - 1 - nh'(z)\frac{\partial\phi'(z^L)}{\partial r}\left[h(z^L)V + \pi - z^L\right] > 0$$

$$g_\pi = -\left[h'(z^L) + nh'(z)\phi'(z^L)\right] < 0$$

$$g_n = \left[h'(z^L)V - 1\right]h(z) - \left[h'(z)\phi'(z^L) + nh'(z)\frac{\partial\phi'(z^L)}{\partial n}\right]\left[h(z^L)V + \pi - z^L\right]$$

$$g_V = h'(z^L)V[r + nh(z)] - nh'(z)\phi'(z^L)h(z^L)V - nh'(z)\frac{\partial\phi'(z^L)}{\partial V}\left[h(z^L)V + \pi - z^L\right]$$

where $f_z < 0$ follows from the assumption of stability and g_n and g_V are the only ambiguous signs. It follows that comparative statics for n and V is ambiguous but $\mathrm{d}z^L/\mathrm{d}r > 0$, $\mathrm{d}z/\mathrm{d}r > 0$, $\mathrm{d}z^L/\mathrm{d}\pi < 0$ and $\mathrm{d}z/\mathrm{d}\pi < 0$. Q.E.D.[1]

Proof of Proposition 2. Notice that:

$$\frac{\partial\left[\sum_{j=1}^{n} h(z^j)\right]}{\partial z^L} = nh'(z)\phi'(z^L) > 0$$

is the only element that differentiates the system (7)–(8) from the system (3)–(4). Since it is positive it is clear that the marginal cost of investment for the leader is higher (since it induces greater investment by the outsiders) and thus the leader invests less for a given investment from the outsiders. Strategic complementarity ($\phi'(z^L) > 0$) implies that the same must be true also for the outsiders in equilibrium, which proves (*a*). A graphical proof of (*b*) is in the text. Q.E.D.

Proof of Proposition 3. To complete the proof we need to show rigorously that the choice of the leader given by (10) is indeed a global maximum, or, in other words, that the option of zero investment is dominated by that choice. If we use the equilibrium free entry condition of the second stage to rewrite the objective function of the leader as:

$$\begin{aligned}\Pi^L &= \frac{h(z^L)V + \pi - z^L}{[r + nh(z) + h(z^L)]} - F \\ &= \frac{h(z^L)V + \pi - z^L}{h(z)V - z}F - F\end{aligned}$$

[1] However, if n is large enough, we always have $\mathrm{d}z^L/\mathrm{d}n > 0$, $\mathrm{d}z/\mathrm{d}n > 0$ and, if n is small enough, we always have $\mathrm{d}z^L/\mathrm{d}V > 0$ and $\mathrm{d}z/\mathrm{d}V > 0$.

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we notice that the local maximum satisfying the first order condition $h'(z^L)V = 1$ is a global maximum if:

$$\frac{h(z^L)V + \pi - z^L}{h(z)V - z}F - F > \frac{\pi}{h(z)V - z}F$$
$$\Leftrightarrow \frac{h(z^L)V - z^L}{h(z)V - z} > 1$$

but this is always true since we know that $h(z^L)V - z^L > h(z)V - z$. Q.E.D.

Proof of Proposition 4. Immediate by total differentiation of (6), (9) and (10). Q.E.D.

Proof of Proposition 5. Under Nash competition the respective first order conditions are:

$$\frac{h'(z^i)V}{\left[r + \sum_{i=1}^{n} h(z^i) + h(z^L)\right]} - \frac{h'(z^i)h(z^i)V}{\left[r + \sum_{i=1}^{n} h(z^i) + h(z^L)\right]^2} = 1$$

$$\frac{h'(z^L)V}{\left[r + \sum_{i=1}^{n} h(z^i) + h(z^L)\right]} - \frac{h'(z^L)[h(z^L)V + \pi]}{\left[r + \sum_{i=1}^{n} h(z^i) + h(z^L)\right]^2} = 1$$

and we assume that the second order conditions are satisfied. The free entry condition is now:

$$\frac{h(z^i)}{z^i}V = \left[r + \sum_{i=1}^{n} h(z^i) + h(z^L)\right]$$

which implies a symmetric equilibrium between the entrants with:

$$h'(z^C)\left(1 - \frac{z^C}{V}\right) = \frac{h(z^C)}{z^C},$$

which implies $z^C < \tilde{z}$, where $\tilde{z}$ is defined such that $h(\tilde{z}) = \tilde{z}h'(\tilde{z})$ and $\tilde{z} > \hat{z}$, and (13) and (14) in the text define z^{LC} and n^C.

Under Stackelberg competition the second stage implies $z^C = z^S$, the number of entrant firms is given by:

$$n^S = \frac{V}{z^S} - \frac{r}{h(z^S)} - \frac{h(z^{LS})}{h(z^S)}$$

and z^{LS} is determined by (15) in the text as long as this is a global maximum. If this were true, the equilibrium condition for the investment of the leader would imply, using (12), that:

$$\begin{aligned} h'(z^{LS}) &= \frac{h(z^S)}{z^S} = \frac{h(z^C)}{z^C} \\ &= h'(z^C)\left(1 - \frac{z^C}{V}\right) < h'(z^C) = h'(z^S). \end{aligned}$$

Since $h''(z^{LS}) < 0$ it follows that $z^{LS} > z^C = z^S > 0$, which also implies $n^S < n^C$ and proves the Proposition. Notice that $z^{LS} > \tilde{z}$ since $z^S = z^C < \tilde{z}$. To see that z^{LS} is indeed a global maximum, notice that this is true if z^{LS} provides more expected profits than the zero investment option, that is if:

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$$\frac{h(z^{LS})V+\pi}{h(z^S)V}z^S - z^{LS} > \frac{h(0)V+\pi}{h(z^S)V}z^S$$
$$\Leftrightarrow \frac{h(z^{LS})}{h(z^S)}z^S - z^{LS} > 0$$
$$\Leftrightarrow \frac{h(z^{LS})}{z^{LS}} > \frac{h(z^S)}{z^S} = h'(z^{LS})$$
$$\Leftrightarrow z^{LS} > \tilde{z}$$

which we have shown to be always the case. Q.E.D.

Proof of Proposition 6. The free entry condition in the second stage is:

$$h(z)V^E - z = F[r + nh(z) + h(z^L)]$$

and following the same steps of the basic model, we obtain the Stackelberg equilibrium with:

$$h'\left(z^S\right)\left(V^E - F\right) = 1,$$

which expresses the investment of the entrants as a function of the value of the duopolistic position for an outsider, and

$$n^S = \frac{V^E}{F} - \frac{z^S}{h(z^S)F} - \frac{r + h(z^{LS})}{h(z^S)}.$$

Hence, the leader chooses:

$$\begin{aligned} z^{LS} &= \operatorname{argmax}\left[\frac{h(z^L)V^W + \pi - z^L + n^S h(z^S)V^L}{[h(z^S)V^E - z^S]} - 1\right]F \\ &= \operatorname{argmax}\left[h(z^L)V^W + \pi - z^L + nh(z^S)V^L\right] \end{aligned}$$

with first order condition:

$$h'(z^L)V^W = 1 - \frac{\partial n^S h(z^S)}{\partial z^L}V^L$$

but $\dfrac{\partial n^S h(z^S)}{\partial z^L} = -h'(z^L)$ so in equilibrium we have:

$$h'(z^L)\left(V^W - V^L\right) = 1.$$

Clearly, condition (16), always implies that $z^{LS} > z^S$. The comparison with the Nash case is analogous to the basic case. Q.E.D.

Proof of Proposition 7. In the free entry equilibrium with Stackelberg competition we have:

$$\begin{array}{ll} h'(z^S)[V(x^S) - F] = 1 & h(z^S)V'(x^S) = 1 \\ h'(z^L)V(x^{LS}) = 1 & h(z^L)V'(x^{LS}) = 1 \end{array}$$

$$n^S = \frac{V(x^S)}{F} - \frac{z^S}{h(z^S)F} - \frac{r + h(z^{LS})}{h(z^S)}$$

and the previous results go through. Part (*a*) is analogous to the basic case. By totally differentiating the above system of two conditions determining z^S and x^S, we have:

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$$\begin{bmatrix} \frac{dz}{dF} \\ \frac{dx}{dF} \end{bmatrix} = \frac{1}{\Delta(F)} \begin{bmatrix} h(z)V''(x) & -h'(z)V'(x) \\ -h'(z)V'(x) & h''(z)[V(x)-F] \end{bmatrix} \begin{bmatrix} h'(z) \\ 0 \end{bmatrix}$$

where $\Delta(F) = h''(z)[V(x) - F]h(z)V''(x) - [h'(z)V'(x)]^2$, hence the comparative statics depends on the sign of $\Delta(F)$. However, $\Delta(0)$ is the determinant of the maximisation problem for the leader, which we assume positive for the solution to be interior. By continuity, there exists a right neighbourhood of $F = 0$ for which $\Delta(F) > 0$ and:

$$\frac{dz}{dF} = \frac{h(z)h'(z)V''(x)}{\Delta(F)} < 0$$

$$\frac{dx}{dF} = -\frac{h'(z)^2 V'(x)}{\Delta(F)} < 0$$

which proves (b). Q.E.D.

Appendix B. Patent Races and Growth

B.1. In this part I show how to solve for the equilibrium variables in the case of Nash competition in the market for innovations. The free entry condition provides a constant effective discount rate:

$$\begin{aligned} r + p^C &= \frac{(\phi z)^\epsilon V - z}{F} = \frac{\left[\phi \epsilon^{\frac{1}{1-\epsilon}} \phi^{\frac{\epsilon}{1-\epsilon}} (V-F)^{\frac{1}{1-\epsilon}}\right]^\epsilon V - \epsilon^{\frac{1}{1-\epsilon}} \phi^{\frac{\epsilon}{1-\epsilon}} (V-F)^{\frac{1}{1-\epsilon}}}{F} \\ &= \frac{(\phi X)^{\frac{\epsilon}{1-\epsilon}} \left[\epsilon^{\frac{\epsilon}{1-\epsilon}} \left(\frac{1-\alpha}{\alpha} - \eta\right)^{\frac{\epsilon}{1-\epsilon}} - \epsilon^{\frac{1}{1-\epsilon}} \left(\frac{1-\alpha}{\alpha} - \eta\right)^{\frac{1}{1-\epsilon}}\right]}{\eta (r+p)^{\frac{\epsilon}{1-\epsilon}}} \\ &= \left[\epsilon\left(\frac{1-\alpha}{\alpha} - \eta\right)\right]^\epsilon \left[\frac{1 - \epsilon\left(\frac{1-\alpha}{\alpha} - \eta\right)}{\eta}\right]^{1-\epsilon}. \end{aligned}$$

Using this to state the expected value of innovation explicitly, we can obtain the equilibrium per firm flow of investment (I drop indexes j since symmetry between intermediate goods holds):

$$\begin{aligned} z_\kappa^C &= \epsilon^{\frac{1}{1-\epsilon}} \phi_\kappa^{\frac{\epsilon}{1-\epsilon}} \left[\left(\frac{1-\alpha}{\alpha}\right) \frac{X(\kappa+1)}{(r+p_\kappa)} - \frac{\eta X(\kappa+1)}{(r+p_\kappa)}\right]^{\frac{1}{1-\epsilon}} \\ &= \epsilon^{\frac{1}{1-\epsilon}} \left(\frac{1-\alpha}{\alpha} - \eta\right) \phi_\kappa^{\frac{\epsilon}{1-\epsilon}} \left[\frac{X(\kappa+1)}{(r+p_\kappa)}\right]^{\frac{1}{1-\epsilon}} \\ &= \frac{\epsilon\eta\left(\frac{1-\alpha}{\alpha} - \eta\right)}{\left[1 - \epsilon\left(\frac{1-\alpha}{\alpha} - \eta\right)\right]} X(\kappa+1) \end{aligned}$$

where I used the fact that $\phi_\kappa = 1/X(\kappa+1)$. This implies that $h(\phi_\kappa z_\kappa^C)$ is a constant and that the value function of the monopolistic position grows at a constant rate:

$$V_\kappa^C = \frac{\pi(\kappa)}{(r+p^c)} = V_{\kappa-1}^C q^{\frac{\alpha}{1-\alpha}}.$$

We can now derive the expected growth rate of the aggregate quality index Q:

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$$g^C = \mathrm{E}\left(\frac{\Delta Q}{Q}\right) = n^c(\phi_\kappa z_\kappa^c)^\epsilon[q^{\alpha/(1-\alpha)} - 1]$$

$$= n^C\left\{\frac{\epsilon\eta\left(\frac{1-\alpha}{\alpha} - \eta\right)}{\left[1 - \epsilon\left(\frac{1-\alpha}{\alpha} - \eta\right)\right]}\right\}^\epsilon [q^{\alpha/(1-\alpha)} - 1]$$

as functions of the number of firms. Using our assumptions and the free entry condition, we derive the equilibrium number of firms n^C as a function of the interest rate. To close the model we equate g^C to the Euler equation for the growth rate of consumption and, making use of expression for n^C, we can solve for the four equilibrium variables, that is the growth rate given in the text, the number of firms in the R&D sector, the arrival rate of innovations and the interest rate.

B.2. In this part I show how to solve for the equilibrium variables in the case of Stackelberg competition in the market for innovations. We want to find the equilibrium values for V_κ^S, z_k^S, z_k^{LS}, g^S, r^S, p^S and n^S (dropping j indexes since symmetry between intermediate goods holds). We can solve the equilibrium by adopting the method of undetermined coefficients. We guess a functional form for the value function:

$$V_\kappa^S = V_{\kappa-1}^S q^{\frac{\alpha}{1-\alpha}} = \psi^S X(\kappa)$$

where ψ^S is a coefficient which must be determined. This implies:

$$z_\kappa^S = \left\{\epsilon\left[\psi^S - \frac{\eta}{(r^S + p^S)}\right]\right\}^{\frac{1}{1-\epsilon}} q^{\frac{\alpha}{1-\alpha}} X(\kappa)$$

$$z_\kappa^{LS} = \left(\epsilon\psi^S\right)^{\frac{1}{1-\epsilon}} q^{\frac{\alpha}{1-\alpha}} X(\kappa).$$

Substituting in the Bellman equation we have:

$$V_\kappa^S = \psi^S X(\kappa) = \frac{\left(\phi z_\kappa^{LS}\right)^\epsilon V_{\kappa+1}^S + \pi(\kappa) - z_\kappa^{LS}}{(r^S + p^S)} - F_\kappa$$

$$= \frac{X(\kappa)}{r^S + p^S}\left(\left[\phi(k) z_\kappa^{LS}\right]^\epsilon \psi^S q^{\frac{\alpha}{1-\alpha}} + \left(\frac{1-\alpha}{\alpha}\right) - \left\{\epsilon\left[\psi^S - \frac{\eta}{(r^S + p^S)}\right]\right\}^{\frac{1}{1-\epsilon}} q^{\frac{\alpha}{1-\alpha}} - \eta q^{\frac{\alpha}{1-\alpha}}\right)$$

or

$$\psi^S = \epsilon^{\frac{\epsilon}{1-\epsilon}} \psi^{S\frac{1}{1-\epsilon}} q^{\frac{\alpha}{1-\alpha}}(1-\epsilon) + \left(\frac{1-\alpha}{\alpha}\right) - \eta q^{\frac{\alpha}{1-\alpha}}.$$

Moreover the equilibrium number of firms and the instantaneous probability of innovation become:

$$n^S = \varepsilon + (1-\epsilon)\frac{V}{F} - \frac{r}{[\epsilon\phi(V-F)]^{\frac{\epsilon}{1-\epsilon}}} - \left(\frac{V}{V-F}\right)^{\frac{1}{1-\epsilon}}$$

$$= \varepsilon + (1-\epsilon)\frac{\psi^S(r^S + p^S)}{\eta} - \frac{r^S}{\left\{\epsilon\left[\psi^S - \frac{\eta}{(r^S + p^S)}\right]\right\}^{\frac{\epsilon}{1-\epsilon}}} - \left[\frac{\psi^S(r^S + p^S)}{\psi^S(r^S + p^S) - \eta}\right]^{\frac{1}{1-\epsilon}}$$

and

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$$p^S = \left[(\phi z^{LS})^\epsilon + n^S(\phi z^S)^\epsilon\right]$$
$$= \epsilon^{\frac{\epsilon}{1-\epsilon}} \left(\frac{1}{r^S + p^S}\right)^{\frac{\epsilon}{1-\epsilon}} \left\{ \left[(r^S + p^S)\psi^S\right]^{\frac{\epsilon}{1-\epsilon}} + n^S\left[\psi^S(r^S + p^S) - \eta\right]^{\frac{\epsilon}{1-\epsilon}} \right\}.$$

Finally we have the two expressions for the growth rate:

$$g^S = \frac{r^S - \rho}{\theta}$$

$$g^S = p^S[q^{\alpha/(1-\alpha)} - 1].$$

The last five equations form the equilibrium system in five unknowns: ψ^S, g^S, r^S, p^S and n^S.

B.3. In this part I show how to solve the social planner problem. In our economy a social planner would decide (1) how much output to consume, (2) how much of it to transform into intermediate goods, (3) how many R&D laboratories (firms) to build for the development of new technologies of each intermediate good, (4) how much to invest for each good and (5) how to share the investment between the laboratories. This choice should maximise utility given the production function, the individual arrival rate function, the fixed cost function and the resource constraint. Here we will provide a heuristic derivation of the social planner solution; for details see Etro (2001).

First of all, it is straightforward to derive from the concavity of the arrival rate that it is optimal to allocate equal flows of investment between all the R&D laboratories. Now, we guess that these flows are linear functions of the future scale of production, let us say $z_{\kappa_j} = \beta X(\kappa_j + 1) = \beta q^{\frac{\alpha}{1-\alpha}} X(\kappa_j)$ for each i in sector j. Obviously the social planner will not distort the choice of the input mix as the monopolistic producer of intermediate goods were doing in the decentralised equilibrium; hence the aggregate quantity produced of intermediate good j can be determined as:

$$X^*(\kappa_j) = \alpha^{1/(1-\alpha)} q^{\kappa_j \alpha/(1-\alpha)}.$$

Substituting the quantity X_j^*, we obtain the resource constraint as:

$$Y^* = \sum_{j=1}^{N} (q^{\kappa_j} X_j^*)^\alpha = \left(\frac{1}{\alpha}\right) \alpha^{1/(1-\alpha)} Q$$
$$= C + \sum_{j=1}^{N} X_j^* \left[1 + n(\beta + \eta) X_j q^{\frac{\alpha}{1-\alpha}}\right] = C + \alpha^{1/(1-\alpha)} Q\left[1 + n(\beta + \eta) q^{\frac{\alpha}{1-\alpha}}\right]$$

from which we can derive an expression for consumption which holds at every instant. Under the optimal allocation of resources, growth is constant and determined by the rate of innovation as:

$$g^* = \sum_{i=1}^{n} \left[\phi(\kappa_j) z_i(\kappa_j)\right]^\epsilon [q^{\alpha/(1-\alpha)} - 1] = n\left(\beta \alpha^{\frac{-1}{1-\alpha}}\right)^\epsilon [q^{\alpha/(1-\alpha)} - 1],$$

using the fact that $\phi_{\kappa_j} = \left[\alpha^{\frac{1}{1-\alpha}} X^*(\kappa_j + 1)\right]^{-1}$ and intertemporal utility can be written as:

$$U = \int_0^\infty \frac{C_t^{1-\theta}}{1-\theta} \mathrm{e}^{-\rho t} \mathrm{d}t = \int_0^\infty \frac{\left(C_0 e^{g^* t}\right)^{1-\theta} \mathrm{e}^{-\rho t}}{1-\theta} \mathrm{d}t = \frac{C_0^{1-\theta}}{(1-\theta)[\rho - (1-\theta)g^*]}.$$

Finally, we can summarise the social planner problem as:

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$$\max_{n,\beta} \frac{\left[\frac{1}{\alpha} - 1 - n(\beta + \eta) q^{\frac{\alpha}{1-\alpha}}\right]^{1-\theta}}{(1-\theta)\left\{\rho - (1-\theta) n \left(\beta \alpha^{\frac{-1}{1-\alpha}}\right)^{\epsilon} [q^{\alpha/(1-\alpha)} - 1]\right\}}.$$

The FOCs for the social planner problem with respect to β and n provide $\beta^* = \frac{\epsilon}{1-\epsilon}\eta$, which implies the optimal level of flow of investment in R&D per firm in the text, and an expression for the optimal number of R&D laboratories n^*.

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