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Technical Appendix to

AMBIGUITY IN PARTNERSHIPS

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Appendix

This appendix contains formal statements of our definitions and results. It concludes with the proofs of the theorems and the corollary.

To Section 3

DEFINITION 1. Let the set of states of nature be S, which we assume to be finite. Denote the set of events (i.e., subsets of S) by $P(S) := \{E \subseteq S\}$. A function $v : P(S) \to [0, 1]$ is a *capacity* if it satisfies $v(\phi) = 0$, $v(S) = 1$, and for any $A, B \subseteq S$ with $A \subseteq B$ we have $v(A) \leq v(B)$.

DEFINITION 2. Let v be a capacity and let $u : S \to \mathbb{R}$ be a function. Let $\pi := (\pi_1, \ldots, \pi_{|S|})$ be a permutation of the states of nature such that $u(\pi_1) \geq u(\pi_2) \geq \ldots \geq u(\pi_{|S|})$. The *Choquet integral* of u with respect to v is

$$\int^C u dv := u(\pi_1) \cdot v(\{\pi_1\}) + \sum_{k=2}^{|S|} u(\pi_k) \cdot [v(\{\pi_1, \ldots, \pi_k\}) - v(\{\pi_1, \ldots, \pi_{k-1}\})].$$

DEFINITION 3. An event $E \subseteq S$ is a *support* of v if $v(S \backslash E) = 0$ and $F \subset E$ implies $v(S \backslash F) > 0$.[1,2]

DEFINITION 4. Let for each player $i \in I$ the capacity $v_i^* : P(S_{-i}) \to [0, 1]$ and the strategy $s_i^* \in S_i$ be given, where $S_{-i} := \Pi_{j \in I \backslash \{i\}} S_j$. The profile $(s_i^*, v_i^*)_{i \in I}$ is an *equilibrium in non-additive beliefs* if for each player i there exists an event E_i^* that is a support of v_i^* such that

$$s_{-i}^* \in E_i^* \subseteq \prod_{j \in I \backslash \{i\}} R_j(v_j^*),$$

[1] There are several competing notions of a support of a capacity. The way the support of a capacity is defined may influence the set of equilibria of the game. For a discussion we refer to Haller (2000).

[2] It should be noted that the support of a capacity may fail to be unique. For examples see Dow and Werlang (1994).

where

$$R_j(v_j^*) := \arg\max_{s_j \in S_j} \int^C u_j(s_j, \cdot)\, dv_j^*,$$

is the set of best responses of player j for given belief v_j^*.

DEFINITION 5. A capacity $v : P(S) \to [0, 1]$ is a *pure belief with centre* $s^* \in S$ if $\{s^*\}$ is the unique support of v. If S is finite, the support of v equals the set of points that have positive mass.

DEFINITION 6. Let $(s_i^*, v_i^*)_{i \in I}$ be an equilibrium such that for each player $i \in I$ the capacity v_i^* is a pure belief with centre s_{-i}^*. Then $(s_i^*, v_i^*)_{i \in I}$ is an *equilibrium in pure beliefs.*

EXAMPLE 1. In general, an equilibrium in pure beliefs may fail to exist, as in the 'matching pennies' game:

	L	R
U	(1,0)	(0,1)
D	(0,1)	(1,0)

For contradiction, suppose that player 1 plays U in an equilibrium in pure beliefs. Then the equilibrium belief of player 2 has its centre at U. It follows that $v_2(\{U\}) > 0$ and $v_2(\{D\}) = 0$. The Choquet expected payoff of player 2 when he chooses L is $v_2(\{D\}) + 0 \cdot [1 - v_2(\{D\})] = 0$. For R it equals $v_2(\{U\}) + 0 \cdot [1 - v_2(\{U\})] > 0$, and so the unique best response of player 2 is R. This implies that, in this equilibrium, the belief of player 1 is pure with centre R. For such belief his unique best response is D and player 1 fails to play U. By a similar argument it follows that in equilibrium player 1 does not play D either and therefore no equilibrium in pure beliefs exists.

To Section 4

Let r be a positive integer and denote $\Delta := \frac{1}{r}$. Let $[\underline{x}, \bar{x}]_\Delta$ denote the set of all rational numbers between $\underline{x}$ and $\bar{x}$, which are integer multiples of Δ. We denote $(\underline{x}, \bar{x})_\Delta$, $[\underline{x}, \bar{x})_\Delta$ and $(\underline{x}, \bar{x}]_\Delta$ accordingly.

Let each agent $k \in N := \{1,\dots, n\}$ choose an effort level from the set $S_k := [0, 2]_\Delta$. The production of the partnership for the effort levels $s := (s_1,\dots, s_n) \in S := \prod_{k \in N} S_k$ is given by

$$\varphi(s) := \sum_{k \in N} s_k.$$

A *sharing rule* is a profile $(g_k)_{k \in N}$ of functions that assign each agent a (possibly negative) share of the production of the partnership, such that the *budget balance* condition holds, i.e., for each $\varphi \in [0, 2n]_\Delta$ we have $\sum_{k \in N} g_k(\varphi) = \varphi$.

In the absence of ambiguity, the utility level obtained by agent k for a given profile of effort levels $s := (s_1,\dots, s_n)$ equals $g_k(\varphi(s)) - c(s_k)$. When he has a non-additive belief v_k, his *decision problem* becomes:

$$\max_{s_k \in S_k} \int^C g_k(\varphi(s_k, s_{-k}))\mathrm{d}v_k - c(s_k).$$

We denote $[t, \rightarrow)_\Delta := [t, 2(n-1)]_\Delta$. For notational convenience, we introduce the capacity $\tilde{v}_k : P([0, \rightarrow)_\Delta) \rightarrow [0, 1]$, which denotes the mass v_k assigns to the event that the total effort of the agents other than k is in some set E. So for each $E \subseteq [0, \rightarrow)_\Delta$ we have

$$\tilde{v}_k(E) := v_k(\{s_{-k} \in S_{-k} | \sum_{j \in N \setminus \{k\}} s_j \in E\}).$$

Assumption 4. For c and Δ we have that

$$\frac{\Delta}{n} < c(1) - c(1 - \Delta).$$

Assumption 4 states that Δ is 'sufficiently small' to allow for a 'sufficiently close' approximation of the effort levels in the continuous case.[3]

To Section 6

For proofs, see the subsequent section of this Appendix.

Theorem 5.[4] In the game induced by the entrepreneurial sharing rule, there exists an equilibrium in pure beliefs $((s_k^*)_{k\in N}, (v_k^*)_{k\in N})$, with

(i) $\Psi_{\tilde{v}_1^*}([\tilde{s}_{-1}, \rightarrow)_\Delta) < 1 - 1/\Delta \cdot [c(1) - c(1-\Delta)]$, where $\tilde{s}_{-1} := \sum_{j=2}^{n} s_j$ and
(ii) $\Psi_{v_j^*}(\{\underline{0}_{-j}\}) > c(1)/(G+1)$ for each $j \in J$, where $\underline{0}_{-j}$ denotes each partner other than j choosing effort level 0.

Moreover, in every equilibrium in pure beliefs with (i) and (ii), all partners provide the efficient levels of effort.

Theorem 6. Let the belief v_1 of agent 1 be pure such that if $s_{-1} \in S_{-1}$ is the centre of v_1 then $\Psi_{\tilde{v}_1}([\tilde{s}_{-1}, \rightarrow)_\Delta) = 0$. Let the belief v_j of each $j \in J$ be restricted to be pure with $\Psi_{v_j}(\{\underline{0}_{-j}\}) > c(1)/(G+1)$. Then every equilibrium is ex-ante efficient.

Proofs To Section 6

Lemma 7. Let $j \in J$ and let v_j represent the pure belief of agent j. Let v_j be such that for the ambiguity level of $\{\underline{0}_{-j}\}$ we have $\Psi_{v_j}(\{\underline{0}_{-j}\}) > c(1)/(G+1)$. Then the (unique) best response of agent j is $s_j^* = 1$.

Proof of Lemma 7

(i) $s_j = 1$.

For $s_j = 1$, we have for the Choquet expected utility of agent j that

$$\int^C f_j(\varphi(\underline{1}_{-j}, s_j))\mathrm{d}v_j - c(s_j) = 1 - c(1),$$

where $\underline{1}_{-j}$ denotes each partner other than j choosing effort level 1.

[3] In particular, this assumption is satisfied whenever $c'(1 - 1/\Delta) \geq \Delta/n$.
[4] In a stronger but less accessible formulation, the theorem generalises to continuous strategy spaces by letting $\Delta \rightarrow 0$, without imposing $\Psi_{\tilde{v}_1^*}([\tilde{s}_1, \rightarrow)) = 0$.

(ii) $s_j < 1$.

For given $s_j < 1$, and $G \geq 0$ we can make the Choquet expected utility of agent j

$$\int^C f_j(\varphi(\underline{1}_{-j}, s_j))\mathrm{d}v_j - c(s_j)$$

smaller than $1 - c(1)$ by choosing $\Psi_{v_j}(\{\underline{0}_{-j}\})$ sufficiently large (but, of course, less than 1). To see this, note that $\Psi_{v_j}(\{\underline{0}_{-j}\}) = 1 - v_j(\{\underline{0}_{-j}\}) - v_j(S_{-j}\backslash\{\underline{0}_{-j}\})$. By definition, v_{-j} is a pure belief, so if its centre is $s_{-j} \neq \underline{0}_{-j}$, we have $v_j(\{\underline{0}_{-j}\}) = 0$ and $v_j(S_{-j}\backslash\{\underline{0}_{-j}\}) = 1 - \Psi_{v_j}(\{\underline{0}_{-j}\})$. Note that:

$$\int^C f_j(\varphi(\underline{1}_{-j}, s_j))\mathrm{d}v_j - c(s_j) \leq [1 - \Psi_{v_j}(\{\underline{0}_{-j}\})] \cdot 1 + [\Psi_{v_j}(\{\underline{0}_{-j}\})] \cdot (-G) - 0.$$

By choosing $\Psi_{v_j}(\{\underline{0}_{-j}\}) > c(1)/(G+1)$, the expected utility of agent j becomes less than $1 - c(1)$. If v_j is pure with centre $s_{-j} = \underline{0}_{-j}$, then $v_j(S_{-j}\backslash\{\underline{0}_{-j}\}) = 0$. It follows that the weight of the outcome $-G$ in the Choquet integral is $\Psi_{v_j}(\{\underline{0}_{-j}\}) = 1 - v_j(S_{-j}\backslash\{\underline{0}_{-j}\}) = 1$, and the Choquet expected utility of agent j is less than $1 - c(1)$.

(iii) $s_j > 1$.

The Choquet expected utility of agent j for $s_j = 1 + t \cdot \Delta > 1$ for any integer $t > 0$ equals, by a similar argument,

$$\int^C f_j(\varphi(\underline{1}_{-j}, 1 + t \cdot \Delta))\mathrm{d}v_j - c(1 + t \cdot \Delta) = 1 - c(1 + t \cdot \Delta) < 1 - c(1).$$

Therefore, $s_j = 1$ is the unique best response of agent j. ■

LEMMA 8. Let the capacity v_1 represent the belief of agent 1. Let v_1 be pure with centre $\underline{1}_{-1}$, where $\underline{1}_{-1}$ denotes each partner other than 1 choosing effort level 1. Let the ambiguity level $\Psi_{\tilde{v}_1}\big([n-1, \rightarrow)_\Delta\big) < 1 - 1/\Delta \cdot [c(1) - c(1-\Delta)]$. Then the (*unique*) best response of agent 1 is $s_1^* = 1$.

Proof of Lemma 8. Since v_1 is a pure belief at $\underline{1}_{-1}$ by assumption, we have that for each $A \in P(S_{-1})$ with $\underline{1}_{-1} \notin A$ that $v_1(A) = 0$. In particular, $v_1(S_{-1}\backslash\{\underline{1}_{-1}\}) = 0$.

Since $\Psi_{\tilde{v}_1}\big([n-1, \rightarrow)_\Delta\big) := 1 - \tilde{v}_1\big([n-1, \rightarrow)_\Delta\big) - \tilde{v}_1\big([0, n-1)_\Delta\big)$ is assumed to be less than $1 - 1/\Delta \cdot [c(1) - c(1-\Delta)]$, we have $\tilde{v}_1\big([n-1, \rightarrow)_\Delta\big) > 1/\Delta \cdot [c(1) - c(1-\Delta)]$.

(i) $s_1 < 1$.

The difference in the Choquet expected utility of agent 1 for choosing the effort level of 1 instead of $s_1 < 1$ is at least

$$\begin{aligned}
&\tilde{v}_1\big([n-1, \rightarrow)_\Delta\big) - c(1) - \tilde{v}_1\big([n-1, \rightarrow)_\Delta\big) \cdot s_1 + c(s_1) \\
&> \frac{1}{\Delta} \cdot [c(1) - c(1-\Delta)] \cdot (1 - s_1) - c(1) + c(s_1) \\
&\geq \frac{1}{\Delta} \cdot [c(1) - c(1-\Delta)] \cdot [1 - (1-\Delta)] - c(1) + c(1-\Delta) = 0
\end{aligned}$$

by using $c'(1) = 1$, the strict convexity of c and the structure of the sharing rule. So $s_1 < 1$ is not a best response.

(ii) $s_1 > 1$.

For this case, we have by $c'(1) = 1$, the strict convexity of c and by the structure of the sharing rule, that the difference in Choquet expected utility for choosing the effort level of $s_1 = 1 + \Delta$ instead of 1 is at most

$$1 + \Delta - c(1 + \Delta) - 1 + c(1) = \Delta + c(1) - c(1 + \Delta) < 0,$$

since we have $c(1 + \Delta) - c(1) > \Delta$.

Therefore, we have that for $\Psi_{\tilde{v}_1}\big([n - 1, \rightarrow)_\Delta\big) < 1 - 1/\Delta \cdot [c(1) - c(1 + \Delta)]$ the unique best response of agent 1 is $s_1 = 1$. ■

Proof of Theorem 5. For each $j \in J$ let v_j^* be a pure belief such that the ambiguity level $\Psi_{v_j^*}(\{\underline{0}_{-j}\}) > c(1)/(G + 1)$. Then $s_j^* = 1$ is the (unique) best response. By Lemma 7, such v_j^* exists.

For every pure belief v_1^* with centre $\underline{1}_{-1}$ some s_1^* exists that maximises

$$\int^C f_1(\varphi(\underline{1}_{-1}, s_1))\mathrm{d}v_1 - c(s_1).$$

The combination $\big((s_k^*)_{k\in N}, (v_k^*)_{k\in N}\big)$ is an equilibrium in the game induced by the sharing rule.

Let (s_k^*, v_k^*) be such an equilibrium.

Since $\Psi_{v_j^*}(\{\underline{0}_{-j}\}) > c(1)/(G + 1)$, it follows from Lemma 8 that for each $j \in J$ we have $s_j^* = 1$. Since (s_k^*, v_k^*) is an equilibrium, v_1^* is a pure belief at $\underline{1}_{-1}$. By assumption of the theorem, $\Psi_{\tilde{v}_1^*}\big([\tilde{s}_{-1}, \rightarrow)_\Delta\big) < 1 - 1/\Delta \cdot [c(1) - c(1 - \Delta)]$, so it follows from Lemma 8, that $s_1^* = 1$. This completes the proof. ■

Proof of Theorem 6. According to Theorem 5 we have in equilibrium for each $k \in N$ that $s_k^* = 1$. So it remains to show that the Choquet expected utility for each agent $k \in N$ equals $1 - c(1)$.

For agent 1, we have

$$\int^C f_1(\varphi(s_{-1}, 1))\mathrm{d}v_1 - c(1) = 1 \cdot \big[\tilde{v}_1^*\big([\tilde{s}_{-1}^*, \rightarrow)_\Delta\big) - v_1^*(\emptyset)\big] - c(1) = 1 - c(1).$$

For any agent $j \in J$ we have

$$\int^C f_j(\varphi(s_{-j}, 1))\mathrm{d}v_j - c(1) = 1 \cdot [v_j^*(S_{-j}) - v_j^*(\emptyset)] - c(1) = 1 - c(1),$$

which proves the theorem. ■

Proof of Corollary 3. Suppose that for each $k \in N$ we have $\tilde{v}_k^* = \tilde{v}^*$. For the equilibrium to be ex-post efficient, the assumption of Theorem 5 requires with respect to agent 1 that $\Psi_{\tilde{v}^*}\big([\tilde{s}_{-1}, \rightarrow)_\Delta\big) < 1 - 1/\Delta \cdot [c(1) - c(1 - \Delta)]$. Since the belief is pure with centre $\underline{1}_{-k}$, this is equivalent to $\tilde{v}^*\big([n - 1, \rightarrow)_\Delta\big) > 1/\Delta \cdot [c(1) - c(1 - \Delta)]$. For the agents in J, we must have $\Psi_{v^*}(\{\underline{0}_{-k}\}) > c(1)/(G + 1)$. Since the beliefs are pure with centre $\underline{1}_{-k}$, this holds whenever $\tilde{v}^*\big((0, \rightarrow)_\Delta\big) < 1 - c(1)/(G + 1)$.

Denote $a_1 := \tilde{v}(\{n - 1\})$ and $a_2 := \tilde{v}\big([n - 1, \rightarrow)_\Delta\big)$. Let $\tilde{v}\big((0, \rightarrow)_\Delta\big) = a_2 < 1 - c(1)/(G + 1)$ and $a_2 > 1/\Delta \cdot [c(1) - c(1 - \Delta)]$. Definition 1 requires that $a_1 = \tilde{v}(n - 1) < \tilde{v}\big((0, \rightarrow)_\Delta\big) = a_2$. For G sufficiently large, such a_1 and a_2 exist.

The pure belief

$$\tilde{v}(E) := \begin{cases} 0 & \text{if } \; n-1 \notin E \\ a_1 & \text{if } n-1 \in E \not\supseteq [n-1, \rightarrow)_\Delta \\ a_2 & \text{if } S \neq E \supseteq [n-1, \rightarrow)_\Delta \\ 1 & \text{if } E = S \end{cases}$$

satisfies both conditions of Theorem 5 and is convex since

$$\tilde{v}([n-1, \rightarrow)_\Delta) + \tilde{v}((0, \rightarrow)_\Delta) \leq \tilde{v}(S) + \tilde{v}(\{n-1\}). \;\blacksquare$$