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Technical Appendix to

COMPETITIVE BIDDING IN AUCTIONS WITH PRIVATE AND COMMON VALUES*

Jacob K. Goeree and Theo Offerman

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Appendix: Proofs

Below we shall need the following properties implied by logconcavity.

LEMMA A1. *If $f(x, y)$ is logconcave then so are its marginals $f(x)$ and $f(y)$.*

Proof. See An (1998).

COROLLARY A1. *The density $f_s(s)$ is logconcave in s.*

Proof. The joint density of v and s is given by $f(v, s) = f_v(v)\, f_c(v/n-s)$. It is straightforward to verify that under Assumption 2 the Hessian of $\log\,[f(v, s)]$ is negative semi-definite, i.e. $f(v, s)$ is logconcave. Lemma A1 thus implies that $f_s(s)$ is logconcave. Q.E.D.

LEMMA A2. *The hazard rate, $h(x) = f(x)/[1-F(x)]$, of a logconcave density, $f(x)$, is everywhere non-decreasing.*

Proof. Let $f(x)$ be logconcave then

$$f'(x)[1-F(x)] = \int_x^{x_H} f'(x)f(t)\mathrm{d}t \geq \int_x^{x_H} f'(t)f(x)\mathrm{d}t \geq -f(x)^2,$$

where the first inequality follows since logconcavity of f implies that $f'(x)/f(x)$ is non-increasing in x, so $f'(x)/f(x) \geq f'(t)/f(t)$ for all $t \geq x$. It is straightforward to rewrite the above equation as: $\{f(x)/[1-F(x)]\}' \geq 0$. Q.E.D.

LEMMA A3. *Let X and Y be random variables. The conditional expectation* $\mathrm{E}(Y|X \leq x)$ *satisfies*

$$\frac{\partial}{\partial x}\mathrm{E}(Y|X \leq x) = [\mathrm{E}(Y|X = x) - \mathrm{E}(Y|X \leq x)]\frac{f_X(x)}{F_X(x)}.$$

Proof. Write the conditional expectation $\mathrm{E}(Y|X \leq x)$ as

$$\mathrm{E}(Y|X \leq x) = \int_{x_L}^{x} \mathrm{E}(Y|X = t)\frac{f_X(t)}{F_X(x)}\mathrm{d}t.$$

Differentiating the right side with respect to x yields the desired result. Q.E.D.

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Proof of Lemma 1. Since $f(v|s) = f_v(v)f_c(v/n-s)/f_s(s)$, logconcavity of $f_c(\cdot)$ is equivalent to

$$\partial_v \partial_s \log[f(v|s)] \geq 0.$$

This means that $\partial_s \log[f(v|s)] = \partial_s f(v|s)/f(v|s)$ is increasing in v, or, $\partial_s f(v_1|s)/f(v_1|s) \geq \partial_s f(v_2|s)/f(v_2|s)$ for all $v_1 \geq v_2$. This inequality can be rewritten as $\partial_s[f(v_1|s)/f(v_2|s)] \geq 0$. Hence, for all $s_1 \geq s_2$ we have $f(v_1|s_1)/f(v_2|s_1) \geq f(v_1|s_2)/f(v_2|s_2)$.[1] Since this is true for all $v_1 \geq v_2$, it remains true if we integrate v_1 from v to v_H and integrate v_2 from v_L to v, which yields $[1-F(v|s_1)]/F(v|s_1) \geq [1-F(v|s_2)]/F(v|s_2)$, or, $F(v|s_1) \leq F(v|s_2)$. In other words, $F(v|s_1)$ first-order stochastically dominates $F(v|s_2)$, which implies $\mathrm{E}(v|s_1) \geq \mathrm{E}(v|s_2)$ for all $s_1 \geq s_2$. So $\mathrm{E}(v|s = x)$ is non-decreasing in x. Lemma A3 applied to $X = s$, $Y = v$ yields: $\partial_x \mathrm{E}(v|s \leq x) = [\mathrm{E}(v|s = x) - \mathrm{E}(v|s \leq x)]f_s(x)/F_s(x)$, which is non-negative since $\mathrm{E}(v|s = x)$ is non-decreasing in x. The proofs for the conditional expectations of the cost are similar. Q.E.D.

Below we use the following notation. The conditional density of y_1 given $y_1 \leq s$ is: $f_{y_1}(x|y_1 \leq s) = f_{y_1}(x)/F_{y_1}(s) = (n-1)f_s(x)F_s(x)^{n-2}/F_s(s)^{n-1}$ for $x \leq s$, the corresponding distribution function is $F_{y_1}(x|y_1 \leq s) = F_s(x)^{n-1}/F_s(s)^{n-1}$.

Proof of Proposition 1. An equivalent way to write (2) is: $B(x) = (n-1)/n\mathrm{E}(v|s \leq x) + \mathrm{E}(y_1|y_1 \leq x)$ and its derivative is easily derived from Lemma A3: $B'(x) = (n-1)/n[\mathrm{E}(v|s=x) - \mathrm{E}(v|s \leq x)]\, f_s(x)/F_s(x) + [x - \mathrm{E}(y_1|y_1 \leq x)]f_{y_1}(x)/F_{y_1}(x)$. The last term is positive (the event $y_1 = x$ occurs with probability zero) and the first term is non-negative by Lemma 1. Hence, $B(\cdot)$ is increasing. Suppose that bidders $2,\ldots,n$ bid according to (2). Bidder 1's expected profit is:

$$\pi_1^e(b) = \left\{ s_1 + \frac{n-1}{n}\mathrm{E}[v|s \leq B^{-1}(b)] - b \right\} F_s^{n-1}[B^{-1}(b)].$$

From Lemma A3, the derivative of the expected profit with respect to b evaluated at $b = B(x)$ is:

$$\pi_1^{e\prime}[B(x)] = \frac{f_{y_1}(x)}{B'(x)}\left[s_1 + \frac{1}{n}\mathrm{E}(v|s = x) + \frac{n-2}{n}\mathrm{E}(v|s \leq x) - B(x) - B'(x)\frac{F_{y_1}(x)}{f_{y_1}(x)} \right].$$

Using the expression for B' derived above it is straightforward to verify that the term in the brackets equals $s_1 - x$. Together with monotonicity of $B(\cdot)$ this shows $B(s_1)$ is bidder 1's unique optimal bid. Q.E.D.

Proof of Proposition 2. Note that (3) is equivalent to: $B(x) = x + (1/n)\,\mathrm{E}(v|s = x) + [(n-2)/n]\,\mathrm{E}(v|s \leq x)$. The last two terms are non-decreasing in x (Lemma 1), so $B(\cdot)$ is increasing. Assume that bidders $2,\ldots,n$ bid according to (3). Bidder 1's expected payoff is:

$$\pi_1^e(b) = \left\{ s_1 + \frac{n-1}{n}\mathrm{E}[v|s \leq B^{-1}(b)] - \int_{s_L}^{B^{-1}(b)} B(x)f_{y_1}[x|s \leq B^{-1}(b)]\mathrm{d}x \right\} F_s^{n-1}[B^{-1}(b)].$$

The second term in the brackets on the right side represents the sum of the values of bidders 2 to n given that their surpluses are less than $B^{-1}(b)$. An equivalent way of writing this term is by choosing one of them to have the highest surplus, $y_1 \leq B^{-1}(b)$, while the $n-2$ other have surpluses less than or equal to y_1. The expected profit then becomes

[1] In other words, $f(v|s)$ satisfies the monotone likelihood property, see Milgrom (1981).

$$\pi_1^e(b) = \int_{s_L}^{B^{-1}(b)} \left[s_1 + \frac{1}{n}\mathrm{E}(v|s=x) + \frac{n-2}{n}\mathrm{E}(v|s\le x) - B(x) \right] f_{y_1}(x)\mathrm{d}x$$

$$= \int_{s_L}^{B^{-1}(b)} (s_1 - x)\mathrm{d}F_{y_1}(x).$$

Together with monotonicity of $B^{-1}(\cdot)$ this proves that the unique optimal bid is $B(s_1)$. Q.E.D.

Proof of Proposition 3. Note that each B_k is strictly increasing in x. Suppose bidders $2,\ldots,n$ bid according to (4). When bidder 1 wins the auction her expected profit is: $s_1 + 1/n\sum_{j=1}^{n-1}\mathrm{E}(v|s = s_{j+1}) - B_{n-2}(s_2)$, where the s_j are the realisations of the others' surpluses arranged in increasing order, i.e. $s_2 \ge \cdots \ge s_n$. Using the definition of B_{n-2} the expected payoff can be written as: $s_1 + (1/n)\ \mathrm{E}(v|s = s_2) - (2/n)\ \mathrm{E}(v|s = s_2) + \mathrm{E}(c|s = s_2) = s_1 - s_2$. So bidder 1's expected profit is positive only when she has the highest surplus, and using $B(\cdot)$ she wins iff $s_1 = Y_1$. Hence, $B(\cdot)$ is the optimal bidding strategy for player 1. Q.E.D.

Proof of Proposition 4. Expected total surplus equals the winner's valuation of the commodity:

$$W = \int_{s_L}^{s_H} \left[x + \frac{n-1}{n}\mathrm{E}(v|s\le x) \right] f_{Y_1}(x)\mathrm{d}x,$$

so $W = \mathrm{E}(s|s = Y_1) + [(n-1)/n]\ \mathrm{E}(v|s \le Y_1) = (1/n)\ \mathrm{E}(v|s = Y_1) + [(n-1)/n]\ \mathrm{E}(v|s \le Y_1) - \mathrm{E}(c|s = Y_1)$. The first two terms represent the expected value of the commodity given that one bidder has the highest surplus and the others have lower surpluses: this is just $\mathrm{E}(V)$. Hence $W = \mathrm{E}(V) - \mathrm{E}(c|s = Y_1)$.

The Envelope Theorem implies that the derivative of the equilibrium profit $\pi^*(s) = \pi^e[B(s)]$ with respect to a bidder's surplus s equals the equilibrium probability of winning. When surpluses are i.i.d. across bidders the equilibrium probability of winning is simply $F_s(s)^{n-1}$ and the equilibrium profit is therefore $\pi^*(s) = \int_{s_L}^{s} F_s(x)^{n-1}\mathrm{d}x$. This is the expected equilibrium profit of a bidder with surplus s in any of the three auctions discussed above. The *ex ante* expected equilibrium *payoffs* accrue to the winning bidder, so

$$\pi_{winner} = n\int_{s_L}^{s_H}\int_{s_L}^{s} F_s^{n-1}(x)\mathrm{d}x\,\mathrm{d}F_s(s) = n\int_{s_L}^{s_H} [1 - F_s(x)]F_s^{n-1}(x)\mathrm{d}x,$$

where the last expression is obtained by changing the order of integration. Partially integrating this expression yields: $\pi_{\mathrm{winner}} = \mathrm{E}(Y_1) - \mathrm{E}(Y_2)$. Finally, revenue is the difference between total surplus and the winner's profit: $R = W - \pi_{\mathrm{winner}} = W - [\mathrm{E}(Y_1) - \mathrm{E}(Y_2)]$. Q.E.D.

Proof of Proposition 5. Let $\alpha_1 > \alpha_2 > 0$. Suppose, without loss of generality, that $s_1(\alpha_1) = \max_{j=1,\ldots,n} s_j(\alpha_1)$ and $s_k(\alpha_2) = \max_{j=1,\ldots,n} s_j(\alpha_2)$. There are two cases to be considered. If $k = 1$, the costs are the same for both αs. If $k \neq 1$, we have: $\alpha_1 v_1/n - c_1 > \alpha_1 v_k/n - c_k$ and $\alpha_2 v_k/n - c_k > \alpha_2 v_1/n - c_1$ (ties occur with probability zero). Dividing the first inequality by α_1 and the second by α_2 and adding the result yields: $c_1/\alpha_1 + c_k/\alpha_2 < c_1/\alpha_2 + c_k/\alpha_1$, or, equivalently, $c_k\ (1/\alpha_2 - 1/\alpha_1) < c_1\ (1/\alpha_2 - 1/\alpha_1)$. Hence $c_1 > c_k$ when $k \neq 1$. In other words, for some realisations of the value and cost signals, the winner's cost is less for $\alpha = \alpha_2$ than for $\alpha = \alpha_1$, while it is the same for other realisations. *A fortiori*, the expected winner's

cost (i.e. the winner's cost averaged over all possible realisations of the value and cost signals) is less for $\alpha = \alpha_2$. So an increase in α lowers total expected surplus. Next we show that an increase in α does not lower bidders' expected profits. Note that the distribution function, $F_{s(\alpha)}(x)$, of $s(\alpha) = \alpha v/n - c$ is given by

$$F_{s(\alpha)}(x) = \int_{v_L}^{v_H} f_v(v)[1 - F_c(\alpha v/n - x)]\mathrm{d}v,$$

from which it follows that $\mathrm{d}F_{s(\alpha)}(x)/\mathrm{d}\alpha = -1/n\, \mathrm{E}[v|s(\alpha) = x]f_{s(\alpha)}(x) \leq 0$, i.e. the distribution $F_{s(\alpha)}(\cdot)$ is stochastically increasing in α. The difference between the first and second order statistic is:

$$\mathrm{E}[Y_1(\alpha)] - \mathrm{E}[Y_2(\alpha)] = n \int_{s_L(\alpha)}^{s_H(\alpha)} F_{s(\alpha)}^{n-1}(x)[1 - F_{s(\alpha)}(x)]\mathrm{d}x.$$

Differentiating this expression with respect to α yields

$$\frac{\mathrm{d}\{\mathrm{E}[Y_1(\alpha)] - \mathrm{E}[Y_2(\alpha)]\}}{\mathrm{d}\alpha} = \frac{1}{n}\{\mathrm{E}[v|s(\alpha) = Y_1(\alpha)] - \mathrm{E}[v|s(\alpha) = Y_2(\alpha)]\}.$$

Lemma 1 together with $Y_1(\alpha) > Y_2(\alpha)$ ensures that the right side is non-negative. Since bidders are no worse off and total expected surplus is lower, the seller's revenue has to be less. Q.E.D.

Proof of Lemma 2.

$$\mathrm{E}_n(c|s = Y_1) = \int_{s_L}^{s_H} \mathrm{E}(c|s = x)\mathrm{d}F_s^n(x),$$

and

$$\mathrm{E}_{n+1}[c|s = Y_1'(\alpha_0)] = \int_{s_L}^{s_H} \mathrm{E}(c|s = x)\mathrm{d}F_s^{n+1}(x).$$

Since $F^{n+1}(\cdot)$ first-order dominates $F^n(\cdot)$ and $\mathrm{E}(c|s = x)$ is non-increasing in x, we have $\mathrm{E}_{n+1}[c|s = Y_1'(\alpha_0)] \leq \mathrm{E}_n(c|s = Y_1)$. Q.E.D.